Computation of the RG Improved Effective Potential*

– Details of the Numerical Analysis –

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ABSTRACT

Recently we have applied the newly proposed renormalization group (RG) improvement procedure of the finite temperature effective potential to the massive ϕ^4 model, and have shown that i) the massive scalar ϕ^4 model has a rich three-phase structure at $T \neq 0$, two of them are not seen in the ordinary perturbative analysis, and that ii) the $O(N)$ symmetric massive ϕ^4 model has a two-phase structure, one of them are not seen in the perturbative analysis. Both results indicate that the new RG improvement procedure is really an efficient procedure that can incorpolate the nonperturbative effect by resumming correctly the dominant large perturbative corrections through the RG technique. To reach the above observations, however, a bit tedious numerical analysis is unevitable, which is the cost to be paid. In this paper we present the details of the numerical analysis; where the trouble in numerical computation may arise, how it can be resolved, and what is the reliability of the numerical computation. The last problem, i.e., reliability of the computation is studied by comparing the results of numerical computation with that of the high temperature expansion of the full formula. We find that the method we have made use of in the numerical computation is quite successful, giving even in the most troublesome cases the three significant figures agreement with the high temperature expansion. Results on the phase structure of the $O(N)$ symmetric massive ϕ^4 model obtained by the numerical analysis are briefly given.

I. Introduction and summary

To investigate the phase structure of relativistic field theories both at zero and finite temperatures an powerful and convenient tool is the effective potential (EP). The trouble inherent in the analytic evaluation of the EP through the perturbative calculation is the renormalization-scheme (RS) dependence[1] of the results, without solving this problem no reliable physical prediction can be made.

Recently simple but efficient procedures for resumming dominant large perturbative

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corrections by using the renormalization group (RG) technique are proposed[2),3)], and have been applied to the massive ϕ^4 model[3),4)] to investigate the phase structure of the model. These analyses have shown that i) the massive scalar ϕ^4 model at $T \neq 0$ has a three-phase structure, two among the three of the phases being not seen in the ordinary perturbative analysis thus having a nonperturbative origin, and that ii) the $O(N)$ symmetric version of the model in the large-N limit has a two-phase structure being different from the simple model, one of the phases being again not seen in the perturbative analysis. These results indicate that the proposed RG improvement procedures are really efficent in resumming correctly the dominant large perturbative corrections, incorpolating properly the nonperturbative effect as well as resolving the RS dependence problem.

To reach the above observations it is necessary to carry out the numerical evaluation together with the analytic analysis of the RG improved full formulas of the EP. The numerical anlysis is a little bit tedious because of the appearence of complex non-elementary functions with integral expressions, whose integrands show superficial divergences. To get the result we must solve this problem and give a rigorous estimate on the reliability of the performed numerical computation.

In this paper we will present details of the numerical analysis of the RG improved full formulas of the EP, namely, the procedure of numerical integration to carry out the numerical analysis with sufficient accuracy, by focussing on the following questions: where the trouble in numerical computation may arise, how it can be overcome, and what is the reliability of the present numerical analysis. Starting from the one-loop calculation of the EP, there appear two troublesome functions of integral expression in the RG improved formula of the EP. One of them has an integrand that may develop logarithmic divergences, causing little troubles. The other function has an integrand that may develop power divergences, and the function itself is defined as the principal part integral of such divergent integrand. Surely this type of principal part integration causes no trouble if performed analytically, but in the numerical computation it actually causes serious trouble and must be handled carefully.

Main conclusion of the present paper is that the method we have made use of in the numerical anlysis, especially the method of principal part integration, is found to be quite successful, giving in the most troublesome cases the three significant figures reliability. Thus we can rely on the result based on the numerical analysis. Several results on the phase structure of $O(N)$ symmetric massive ϕ^4 model at $T \neq 0$ are presented as an illustration.

This paper is organized as follows. In chapter II all the formulas and expressions that appear in the present analysis of the EP of $O(N)$ symmetric massive ϕ^4 model at $T \neq 0$ are properly defined and given. The method and procedure of the numerical computation and analysis are explained in chapter III. The estimate on the reliability

of the performed numerical computation is also presented in chapter III, where some comparisons of the result based on the numerical computation with that based on the high temperature expansion analysis are given. Last chapter IV is devoted to discussion and comments.

II. Effective potential of the $O(N)$ symmetric massive ϕ^4 model at $T \neq 0$

In this paper we confine our interest in the one-loop evaluation of the effective potential (EP) of $O(N)$ symmetric massive ϕ^4 model at $T \neq 0$ in the large-N limit:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{8}\lambda(\phi^2)^2 - hm^4 , \qquad (1) \\ \phi^2 &= \phi^a\phi^a, \quad a = 1,\ 2,\ \cdots,\ N . \end{aligned}$$

All the starting calculations are performed in the $T = 0$ renormalization scheme, where all the renormalization constants are the same as those of the vacuum theory. The one-loop calculation gives the EP as

$$V_1 = V^{(0)} + V^{(1)} \qquad (2)$$

$$V^{(0)} = \frac{NM^4}{2\lambda} \qquad (3)$$

$$\begin{aligned} V^{(1)} &= \frac{NM^4}{2\lambda}\left[\tau + \lambda\left\{-\frac{b}{4} + \frac{T^4}{\pi^2 M^4}L_0\left(\frac{T^2}{M^2}\right) - \frac{T^2}{2\pi^2 M^2}L_1\left(\frac{T^2}{M^2}\right)\right\}\right] \\ &\quad + m^4\left(h - \frac{N}{2\lambda}\right) \qquad (4) \end{aligned}$$

where $M^2 \equiv m^2 + (1/2)\lambda\phi^2$, $b = 1/16\pi^2$ and

$$\tau \equiv \lambda\left\{\frac{b}{2}\left(\ln\frac{M^2}{\mu^2} - 1\right) + \frac{T^2}{2\pi^2 M^2}L_1\left(\frac{T^2}{M^2}\right)\right\} , \qquad (5)$$

$$L_0\left(\frac{1}{a^2}\right) \equiv \int_0^\infty k^2\, dk\, \ln[1 - \exp\{-\sqrt{k^2 + a^2}\}] , \qquad (6)$$

$$L_1\left(\frac{1}{a^2}\right) \equiv \frac{1}{2}\int_0^\infty \frac{k^2\, dk}{\sqrt{k^2 + a^2}} \frac{1}{\exp\{\sqrt{k^2 + a^2}\} - 1} . \qquad (7)$$

To this order, the RG improvement can be carried out analytically, giving the RG improved EP as

$$\begin{aligned} \bar{V}_1 &= \frac{N\bar{M}^4(t)}{2\bar{\lambda}(t)}\left[1 + \bar{\lambda}(t)\left\{-\frac{b}{4} + \frac{T^4}{\pi^2\bar{M}^4(t)}L_0\left(\frac{T^2}{\bar{M}^2(t)}\right) - \frac{T^2}{2\pi^2\bar{M}^2(t)}L_1\left(\frac{T^2}{\bar{M}^2(t)}\right)\right\}\right] \\ &\quad + \bar{m}^4(t)\left(\bar{h}(t) - \frac{N}{2\bar{\lambda}(t)}\right), \qquad (8) \end{aligned}$$

where

$$\bar{M}^2(t) = \frac{M^2}{1 - tb\lambda}, \quad \bar{\lambda}(t) = \frac{\lambda}{1 - tb\lambda}, \quad \bar{m}^4(t)\left(\bar{h}(t) - \frac{N}{2\bar{\lambda}(t)}\right) = m^4\left(h - \frac{N}{2\lambda}\right), \qquad (9)$$

and all the barred quantities must be evaluated at such a t satisfying

$$\bar{\tau}(t) = \bar{\lambda}\left\{\frac{b}{2}\left(\ln\frac{\bar{M}^2}{\mu^2 e^{2t}} - 1\right) + \frac{T^2}{2\pi^2\bar{M}^2}L_1\left(\frac{T^2}{\bar{M}^2}\right)\right\} = 0. \tag{10}$$

In the present approximation, i.e., in the $N \to \infty$ limit of the model, the field ϕ is free from renormalization, thus $\bar{\phi} = \phi$. As for the details of the RG improvement procedure, see Refs.2)~5). It is worth noticing that the key condition of the RG improvement procedure, $\bar{\tau}(t) = 0$, which chooses the RS so as to minimize the RS dependence, gives in fact the mass gap equation in the self-consistent improvement method[6].

To investigate the structure of the RG improved EP, we must carry out the numerical computation of the above formulas, especially of the function L_1, Eq.(7), which needs a careful and tedious procedure for the principal part integration.

In the region where $\bar{M}^2/T^2 \ll 1$, we can carry out the high temperature expansion which gives different results depending on the sign of $a^2 \equiv \bar{M}^2/T^2$:

(i) $a^2 \equiv \bar{M}^2/T^2 > 0$

$$L_0\left(\frac{1}{a^2}\right) = -\frac{\pi^4}{45} + \frac{\pi^2}{12}a^2 - \frac{\pi}{6}a^3 - \frac{1}{16}a^4\left(\ln\frac{a}{4\pi} + \gamma_E - \frac{3}{4}\right) + \cdots, \tag{11}$$

$$L_1\left(\frac{1}{a^2}\right) = \frac{\pi^2}{12} - \frac{\pi}{4}a - \frac{1}{8}a^2\left(\ln\frac{a}{4\pi} + \gamma_E - \frac{1}{2}\right) - \frac{\zeta(3)}{64\pi^2}a^4 + \cdots, \tag{12}$$

thus giving

$$\bar{\tau} = \bar{\lambda}\left[-tb + \frac{1}{24a^2} - \frac{1}{8\pi a} + \frac{bA}{2} + \cdots\right], \tag{13}$$

$$\frac{\bar{V}_1}{NT^4} = \frac{1}{2\lambda}\left(1 - \frac{\lambda bA}{2}\right)a^4 + \frac{1}{24\pi}a^3 + \cdots, \tag{14}$$

$$A \equiv 2\left(\ln\frac{4\pi T}{\mu} - \gamma_E\right). \tag{15}$$

(ii) $a^2 \equiv \bar{M}^2/T^2 \equiv -c^2 < 0$

$$\mathrm{Re}L_0\left(\frac{1}{a^2}\right) = -\frac{\pi^4}{45} - \frac{\pi^2}{12}c^2 - \frac{1}{16}c^4\left(\ln\frac{c}{4\pi} + \gamma_E - \frac{3}{4}\right) + \cdots, \tag{16}$$

$$\mathrm{Im}L_0\left(\frac{1}{a^2}\right) = -\frac{\pi}{6}c^3 + \frac{\pi}{32}c^4, \tag{17}$$

$$\mathrm{Re}L_1\left(\frac{1}{a^2}\right) = \frac{\pi^2}{12} + \frac{1}{8}c^2\left(\ln\frac{c}{4\pi} + \gamma_E - \frac{1}{2}\right) - \frac{\zeta(3)}{64\pi^2}c^4 + \cdots, \tag{18}$$

$$\mathrm{Im}L_1\left(\frac{1}{a^2}\right) = -\frac{\pi}{16}c^2, \tag{19}$$

thus giving

$$\bar{\tau} = \bar{\lambda}\left[-tb - \frac{1}{24c^2} + \frac{bA}{2} + \cdots\right], \tag{20}$$

$$\frac{\bar{V}_1}{NT^4} = \frac{1}{2\lambda}\left(1 - \frac{b\lambda A}{2}\right)c^4 + \cdots - \frac{c^3}{12\pi}i. \tag{21}$$

The high temperature expansion greatly simplifies the numerical analysis, and the only problem in this case is the accuracy of the approximation.

III. Procedure of the numerical analysis

In this chapter we explain in detail the procedure of numerical computation we have carried out. To get the good estimate on the reliability of numerical integration the use of carefully coded program written in the Fortran language might be preferable. For the present purpose, however, we find that the use of Mathematica is enough and suitable. The main reason is in the present case we have a good approximation method, i.e., the high temperature expansion which is proved to represent the original full formula with high accuracy in an appropriate range of the parameter in the theory, thus we can compare both results in the region where the high temperature expansion is reliable in high accuracy level.

III-a. Where does the trouble in numerical computation arise?

As mentioned before, the only function that really causes trouble in the numerical computation is the function L_1, Eq.(7). As can be easily seen, however, even in L_1 when the argument of the function $a^2 \equiv \bar{M}^2/T^2$ is positive (namely, $\bar{M}^2 > 0$) the integrand of the integral expresssion is a smooth function, causing any trouble in the numerical integration. When the argument of the function $a^2 \equiv \bar{M}^2/T^2$ becomes negative (namely, $\bar{M}^2 < 0$) the integrand develops a superficial power divergence, thus we must devide the integration range and carefully carry out the principal part integration ($c^2 \equiv -a^2$):

$$L_1\left(\frac{1}{a^2}\right) = \frac{1}{2}\int_0^\infty \frac{x^2 dx}{\sqrt{x^2-c^2}} \frac{1}{\exp\{\sqrt{x^2-c^2}\}-1}, \tag{22}$$

$$= \frac{c^2}{2}\left[\int_{1+w}^\infty + \mathrm{PP}\int_{1-w}^{1+w} + \int_0^{1-w}\right] \frac{y^2 dy}{\sqrt{y^2-1}} \frac{1}{\exp\{c\sqrt{y^2-1}\}-1}, \tag{23}$$

where

$$\mathrm{PP}\int_{1-w}^{1+w} = \lim_{\epsilon\to+0}\left[\int_{1-w}^{1-\epsilon} + \int_{1+\epsilon}^{1+w}\right]. \tag{24}$$

The function L_0 has also in its integrand a superficial divergence of logarithmic nature. As is well known logarithmic divergence in the integrand causes any trouble in the numerical computation, especially in the use of Mathematica. Thus the only trouble in the numerical analysis appeared in the present investigation is the one arisen in the function L_1 explained above.

III-b. Prescription of the principal part integration

In order to perform the principal part integration numerically we must get rid of the evaluation of the integrand function at and very near the point of divergence. This can be achieved by the limiting procedure:

$$\mathrm{PP}\int_{1-w}^{1+w}\mathrm{Re}\left[\frac{y^2dy}{\sqrt{y^2-1}}\frac{1}{\exp\{c\sqrt{y^2-1}\}-1}\right]=$$

$$\lim_{\epsilon\to+0}\left[\int_{\sqrt{\epsilon}}^{\sqrt{w}}\frac{2(1+z^2)}{\sqrt{2+z^2}}\frac{dz}{\exp\{zc\sqrt{2+z^2}\}-1}+\int_{\sqrt{w}}^{\sqrt{\epsilon}}\frac{(1-z^2)dz}{\sqrt{2-z^2}}\frac{\sin\{cz\sqrt{2-z^2}\}}{1-\cos\{cz\sqrt{2-z^2}\}}\right]. \quad (25)$$

By choosing small w value appropriately, we can get the numerical estimate of the function L_1 at various values of its argument.

III-c. Reliability of the numerical analysis and the comparison with the high temperature expansion analysis

Here we present in Table 1 the result of the numerical computation of the function L_1 with several choices of small w-values at various negative values of its argument $a^2(\equiv -c^2)$. Results at several positive values of the argument a^2 are also given for comparison. In the same Table also presented are the function L_1 evaluated in the approximation through the high temperature expansion. By noticing that as $c^2(\equiv -a^2)$

Table 1: The results of the numerical computation of the function $\mathrm{Re}L_1$. w is the parameter that specifies the range of the principal part integration, see Eq.(25).

$\bar{M}^2/T^2$	Numerical Value		High T Expansion*	
	$w=10^{-4}$	$w=10^{-5}$	$O(c^4)$	$O(c^6)$
1	0.344736		0.343795	0.341892
0.1	0.749873		0.749873	0.749873
0.01	0.814701		0.814701	0.814701
0.001	0.821683		0.821683	0.821683
0.0001	0.822389		0.822389	0.822389
- 0.0001	0.822378	0.822379	0.822379	0.822379
- 0.001	0.821730	0.821729	0.821729	0.821729
- 0.01	0.816549	0.816532	0.816522	0.816522
- 0.1	0.777742	0.777522	0.777403	0.777422

*$c^2=|\bar{M}^2|/T^2$

becomes smaller the estimate on the basis of the high temperature expansion becomes more accurate, we can check the reliability of our numerical computation. In fact, at the very small value of the argument, say, $c^2 = 0.0001$ where reliability of the high temperature expansion is quite strict, our numerical estimate agrees with that of the high temperature approximation up to five significant figures. The result at $a^2 \equiv \bar{M}^2/T^2 = 1$ is presented to give a rough idea on the "accuracy" of the high temperature expansion which is in the strict sense almost meaningless mathematically at this value of a^2.

With these observations in hand we can safely say that the present numerical computation is quite successful and thus the results obtained through the present numerical analysis correctly represent the physics described by the RG improved EP in the massive ϕ^4 model at $T \neq 0$ under consideration.

IV. Discussion and comments

First let us briefly present the results on the phase structure of the $O(N)$ symmetric massive ϕ^4 model at $T \neq 0$ in the large-N limit obtained through the present numerical analysis. By studying the "exact" mass gap equation (or the RS fixing condition) $\bar{\tau}(t) = 0$ itself we get the $\phi^2 - \bar{M}^2$ relation shown in Figure 1, indicating the exsistence of two phases: the phase I appears in the high temperature expansion analysis, while the phase II is the new phase that may have truely non-perturbative origin. In the second phase II the effective coupling becomes strong and the effective mass-squared also becomes very heavy, indicating this phase to be almost temperature independent super massive strong coupling phase. The first phase I shows a familiar behavior: at low temperature the EP develops a symmetry broken vacuum, and as the temperature becomes higher

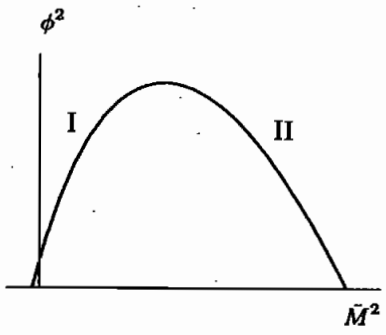

Figure 1: The $\phi^2 - \bar{M}^2$ relation from the "exact" mass gap equation, $\bar{\tau} = 0$. The phase II dose not appear in the perturbative as well as in the high temperature expansion analyses.

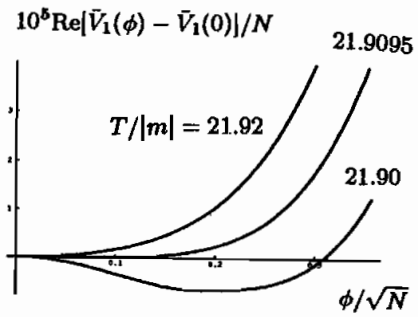

Figure 2: The RG improved effective potential in the phase I at three temperatures.

the minimum of the EP eventually tends to the origin and at some critical temperature symmetry is restored and at sufficiently high temperature the symmetric vacuum at the origin survives, see Figure 2. The phase transition in this model proceeds through the second order transition.

Same analysis can be made in the simple massive scalar ϕ^4 model at $T \neq 0$, through which we can see[4),5)] that the phase structures of both models are completely different from each other. The simple massive scalar ϕ^4 model at $T \neq 0$ has a three-phase structure, only one of the phases can be seen in the ordinary perturbative analysis. The temperature dependent phase transition in this model is strongly the first order, compared to the second order transition in the $O(N)$ symmetric model. Also surprising is the outcome from the simple massive scalar ϕ^4 model at zero temperature[4),5)]. The $T \to 0$ limit of this massive scalar ϕ^4 model at $T \neq 0$ has a stable three-phase structure just as in the model at $T \neq 0$, and does not coinside with the same model at $T = 0$, which is found to be unstable by having a phase unbounded from below.

In the $O(N)$ symmetric ϕ^4 model at $T \neq 0$, $\bar{M}^2$ can become negative (which is the origin of the troublesome numerical computation), in which case the improved EP becomes a function of complex number. The meaning of its imaginary part is not so clear, but in the phase I without passing this region of $\bar{M}^2$ we can not reach the small ϕ region, especially the origin $\phi = 0$, see Figure 1. Because we have started from the one-loop perturbative calculation and performed the RG improvement, only the simple types of diagrams are properly resummed[4),5)] and we must be careful about the reliability problem in the small ϕ region[7)].

The problem of reliability of the analysis in the small ϕ regions is surely an important one to be discussed in detail. This problem, however, is not related with the present issue, i.e., the accuracy of the numerical computation, thus is discussed in the separate paper[4),5)].

Acknowledgments

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References

1) See, e.g., T. Muta, *Foundations of Quantum Chromodynamics* (World Scientific, Singapore, 1987).

2) M. Bando, T. Kugo, N. Maekawa and H. Nakano, *Phys. Lett. B* 301 (1993) 83.

3) H. Nakkagawa and H. Yokota, *Mod. Phys. Lett A* 11 (1996) 2259;
H. Nakkagawa and H. Yokota, *Memoirs of Nara University* 24 (1996) 65.

4) H. Nakkagawa and H. Yokota, "Effective Potential at Finite Temperature – RG Improvement vs. High Temperature Expansion –", Talk given at the YITP International Workshop on Physics of Relativistic Heavy Ion Collisions, Kyoto, Japan, 9-11 June 1997, to be published in *Prog. Theor. Phys. Suppl.* (proceeding volume).

5) H. Nakkagawa and H. Yokota, papers to appear (1998).

6) G. Amelino-Camelia and S.-Y. Pi, *Phys. Rev. D* 47 (1993) 2356.

7) G. Amelino-Camelia, *Phys. Rev. D* 49 (1994) 2740.