論　文

Analysis of the Phase Structure of Thermal QED/QCD through the HTL Improved Ladder Dyson-Schwinger Equation

—— *On the Gauge Dependence of the Solution* ——

Hisao NAKKAGAWA,*) Hiroshi YOKOTA**) and Koji YOSHIDA***)

Institute for Natural Sciences, Nara University, Nara 631-8502, Japan

We solved with a numerical procedure the HTL improved ladder DS equation for the retarded fermion self-energy function Σ_R to study the spontaneous generation of fermion mass in thermal QCD/QED, and studied the gauge-dependence of the solution within a general covariant gauge where the gauge parameter ξ is any constant number.

With the numerical solutions thus obtained, we found the followings; i) The fermion wave function renormalization function $A(P)$ always deviates largely from unity even at the momentum where the mass is defined, thus the corresponding solutions explicitly contradict with the Ward-Takahashi identity. ii) As a result, the obtained solutions strongly depend on the choice of gauge parameters: the critical temperatures and the critical coupling constants significantly change gauge by gauge. In all gauges we studied in the present analysis, we could not find any solution, having a possibility to be consistent with the Ward-Takahashi identity. Thus we are forced to investigate the procedure to find a gauge which enables us to get a solution being consistent with the Ward-Takahashi identity, otherwise we can not obtain any physically meaningful conclusions through the analysis of the point-vertex ladder DS equation no matter how the gauge propagator gets improved.

§1. Introduction and summary

Recent studies on gauge field theory have been revealing rich aspects of phase structure of the matter according to the variation of number density and/or temperature. However, it is supposed to be hard to obtain more detailed understanding of the mechanism of phase transition by means of such theoretical studies, since most of them were carried out on the basis of perturbative calculation or of numerical lattice simulation. This motivated us to survey the problem of phase transition using the Dyson-Schwinger (DS) equation, firstly because the DS equation is derived exactly in the field theory and is the fundamental equation to investigate nonperturbative phenomena within the framework of field theory, and secondly because we can obtain successively improved solutions by the successive refinement of the analytical approximation to its integration kernel, thus revealing the essential contribution that controls the phase transition depending on the temperature/density.

We started our analysis with the DS equation for the retarded fermion mass function Σ_R to study the spontaneous generation of fermion mass in thermal QED/QCD.[1)] In the analysis we used an improved ladder interaction kernel obtained analytically through the Hard-Thermal-Loop (HTL) resummation procedure, in which the ladder kernel is improved by use of the HTL resummed form of the gauge boson propagator.

*) E-mail: nakk@daibutsu.nara-u.ac.jp
**) E-mail: yokotah@daibutsu.nara-u.ac.jp
***) E-mail: yoshidak@daibutsu.nara-u.ac.jp

2009年9月3日受理　*),**) 教養部教授　***) 教養部准教授

We realized that the results obtained[2] are significantly different from those obtained in the preceding analyses with the simple ladder DS equation.[3] It is worth noticing that, in all preceding DS equation analyses including ours, the bare (point) vertex ladder approximation for the integration kernel has been used, thus that any vertex correction does not result.

We here summarize the essential points of our analysis that may give the result significantly different from those of the preceding works. (For details of our results, see Ref. 2).)

i) The DS equation for the retarded fermion mass function Σ_R is derived correctly without any specific assumption for its form.

ii) Nontrivial imaginary parts of the invariant functions A, B and C (see, the definition of Σ_R, Eq. (1·1)) are taken into account.

iii) We adopted the improved ladder integration kernel analytically obtained in the HTL approximation.

Despite the improvements in taking into account correctly a) the (unstable) thermal quasiparticle nature of the fermion in the heat bath and b) the dominant effect of thermal fluctuations through the HTL resummation, it was suggested that the obtained result showed a serious problem, i.e., dependence on the choice of the gauge used in the analysis. We can recognize it to see that the identity $Z_1 = Z_2$ implied by the Ward-Takahashi identity, does not hold, where Z_1 and Z_2 denote the vertex and the wave-function renormalization constants, respectively.

Surely even if some solutions satisfy the identity $Z_1 = Z_2$, it is still possible that they do not satisfy the full Ward-Takahashi identities, and thus they are gauge-dependent solutions. Important point, however, is that those solutions not satisfying $Z_1 = Z_2$ cannot be gauge invariant, namely, that to satisfy the identity $Z_1 = Z_2$ is the necessary condition*) for the obtained solution to be gauge-invariant. It is here worth noting the fact that in thermal QCD in the HTL approximation the Ward-Takahashi identities are formally identical to those satisfied at tree level.[4]

The retarded fermion mass function can be parameterized with the three invariant functions A, B and C as

$$\Sigma_R(P) = (1 - A(P))p_i\gamma^i - B(P)\gamma^0 + C(P) \tag{1·1}$$

The inverse of $A(P)$ at the momentum where the fermion mass is calculated is nothing but the wave function renormalization constant Z_2. As noted above, the vertex renormalization constant Z_1 is exactly unity, $Z_1 = 1$, in our analysis because of the bare (point) vertex ladder approximation for the integration kernel, which is also the case with other works carried out before. It follows that the Ward-Takahashi identity, the statement of gauge invariance, requires $Z_2 = 1$, namely, $A(P) = 1$ at least at the momentum where the fermion mass is calculated. If $A \neq 1$ in the obtained results, it means that the results do not satisfy gauge invariance and hence there is little physical meaning in the results obtained.

*) In the case of QED it is exactly the necessary condition. In the case of QCD it is so in analyzing, as in most of the analyses, solely the decoupled ladder DS equation for Σ_R alone, with the ladder interaction kernel, irrespective of free or HTL-improved gauge boson propagator.

For the vacuum ($T = 0$) case, fortunately, $A(P) = 1$ is verified to hold in the analysis in the Landau gauge.[5] Therefore it is supposed that the analysis of the DS equation in the Landau gauge, apart from the reliability of the ladder approximation, has some physical significance in the quantitative as well as qualitative sense.[6),7] At finite temperatures, however, there is no guarantee that $A = 1$ holds even at the momentum where the fermion mass is calculated. In fact our analysis in the Landau gauge shows that A largely deviates from 1 and becomes even complex number.[2] Namely the analysis of DS equation in the ladder approximation at finite temperature/density QED/QCD is obviously inconsistent with gauge invariance.

This gauge-dependence problem must be tackled seriously in order to draw a definite conclusion from our analysis of the HTL improved ladder DS equation, which indicated the importance to correctly take the dominant effect of thermal fluctuations into the integration kernel through the HTL resummation.

To see the problem more clearly we must clarify how sensitive the obtained results are to the choice of gauges. It will help us to study whether there exists a solution of the DS eqution in the ladder approximation that satisfies $Z_1 = Z_2$ implied by the Ward-Takahashi identity, at finite temperature/density, and also to investigate, if such a solution exists, what is the real difference of it from the "gauge-dependent" solutions.

To answer the questions, in this paper we will solve the HTL improved ladder DS equation in a general covariant gauge and study the dependence of the solutions on the various choices of the gauge parameter, then investigate the possibility to obtain a solution consistent with the identity $Z_1 = Z_2$, which is nothing but the necessary condition for the solution to be gauge-independent (see, the footnote in previous page). We also make an improvement in estimating the numerical integration over the singular part of integration kernel in the present analysis.

We here present the results of our analysis; We find, within gauges where the gauge parameter ξ is constant numbers, i) that the fermion wave function renormalization function $A(P)$ always deviates largely from unity even at the momentum where the mass is defined, thus that the corresponding solutions explicitly contradict with the Ward-Takahashi identity, and ii) that, as a result, the obtained solutions strongly depend on the choice of gauge parameters: the critical temperatures and the critical coupling constants significantly change gauge by gauge. In all gauges we study in the present analysis, we can not find any solution that has a possibility to be consistent with the identity $Z_1 = Z_2$. Thus we are forced to investigate the procedure to find a gauge which enables us to get a solution being consistent with the Ward-Takahashi identity, otherwise we can not obtain any physically meaningful conclusions through the analysis of the point-vertex ladder DS eqution no matter how the gauge propagator gets improved.

This paper is organized as follows; In the next §2 we present the HTL resummed Dyson-Schwinger equation for the retarded fermion self-energy function Σ_R, and explain the improved ladder approximation adopted in the present analysis. Section 3 is devoted to presenting the solutions of the HTL improved ladder DS equation, revealing the large gauge-dependence between solutions. Conclusions of the present paper and the related discussion are given in the last §4. In the Appendix we give some technical details in the numerical analysis in solving the DS equations.

§2. The HTL resummed improved ladder Dyson-Schwinger equation

In this section we present the HTL resummed DS equation for the retarded fermion self-energy function Σ_R. We also give an explication about the improved ladder approximation we make use of to the HTL resummed gauge boson propagator.

We then present the HTL resummed improved ladder Dyson-Schwinger equation for the independent invariant scalar functions A, B and C. We also calculate the effective potential for the retarded fermion propagator S_R in order to find the "true solution" when we get several "solutions" of the DS equation.

2.1. *The HTL resummed Dyson-Schwinger equation for the retarded fermion self-energy function Σ_R*

In the real time closed time-path formalism, we obtain, in the massless thermal QED/QCD in the HTL approximation, the DS equation for retarded fermion self-energy function Σ_R:

$$
\begin{aligned}
-i\Sigma_R(P) = -i\Sigma_{RA}(-P,P) = &-\frac{e^2}{2}\int\frac{d^4K}{(2\pi)^4} \\
&\times\left[{}^*\Gamma^{\mu}_{RAA}(-P,K,P-K)S_{RA}(K)\,{}^*\Gamma^{\nu}_{RAA}(-K,P,K-P)\,{}^*G_{RR,\mu\nu}(P-K)\right. \\
&\left.+\,{}^*\Gamma^{\mu}_{RAA}(-P,K,P-K)S_{RR}(K)\,{}^*\Gamma^{\nu}_{AAR}(-K,P,K-P)\,{}^*G_{RA,\mu\nu}(P-K)\right]\ ,
\end{aligned}
\tag{2·1}
$$

Here ${}^*G^{\mu\nu}$ is the HTL resummed gauge boson propagator, whose retarded ($R\equiv RA$) and correlation ($C\equiv RR$) components are given by[8)]

$$
\begin{aligned}
{}^*G^{\mu\nu}_R(K)\equiv&{}^*G^{\mu\nu}_{RA}(-K,K) \\
=&\frac{1}{{}^*\Pi^R_T(K)-K^2-i\epsilon k_0}A^{\mu\nu}+\frac{1}{{}^*\Pi^R_L(K)-K^2-i\epsilon k_0}B^{\mu\nu}-\frac{\xi}{K^2+i\epsilon k_0}D^{\mu\nu},
\end{aligned}
\tag{2·2}
$$

$$
{}^*G^{\mu\nu}_C(K)\equiv{}^*G^{\mu\nu}_{RR}(-K,K) = (1+2n_B(k_0))\left[{}^*G^{\mu\nu}_R(K)-{}^*G^{\mu\nu}_A(K)\right], \tag{2·3}
$$

$$
n_B(k_0)=\frac{1}{\exp(k_0/T)-1}, \tag{2·4}
$$

with ${}^*\Pi^R_T$ and ${}^*\Pi^R_L$ being the HTL contributions to the transverse and longitudinal modes of the retarded gauge boson self-energy, respectively.[9)] The parameter ξ is the gauge-fixing parameter ($\xi=0$ in the Landau gauge). In the above, $A^{\mu\nu}$, $B^{\mu\nu}$ and $D^{\mu\nu}$ are the projection tensors given by[8)]

$$
A^{\mu\nu}=g^{\mu\nu}-B^{\mu\nu}-D^{\mu\nu},\quad B^{\mu\nu}=-\frac{\tilde{K}^\mu\tilde{K}^\nu}{K^2},\quad D_{\mu\nu}=\frac{K^\mu K^\nu}{K^2}, \tag{2·5}
$$

where $\tilde{K}=(k,k_0\hat{\mathbf{k}})$, $k=\sqrt{\mathbf{k}^2}$ and $\hat{\mathbf{k}}=\mathbf{k}/k$ denotes the unit three vector along $\mathbf{k}$.

We use $S(-P,P)\equiv S(P)$ to denote the full fermion propagator, whose retarded

($R \equiv RA$) and correlation ($C \equiv RR$) components are given by

$$S_R(P) \equiv S_{RA}(-P,P) = \frac{1}{\not{P} + i\epsilon\gamma_0 - \Sigma_R}, \tag{2·6}$$

$$S_C(P) \equiv S_{RR}(-P,P) = (1 - 2n_F(p_0))\left[S_R(P) - S_A(P)\right], \tag{2·7}$$

$$n_F(p_0) = \frac{1}{\exp(p_0/T) + 1}, \tag{2·8}$$

with the retarded fermion self-energy function Σ_R decomposed as Eq. (1·1) in terms of the independent invariant scalar functions $A(P)$, $B(P)$ and $C(P)$.

Finally, the HTL resummed 3-point fermion-gauge boson vertex functions, $^*\Gamma^\mu$, are given by

$$\begin{aligned} ^*\Gamma^\mu_{\alpha\beta\gamma} &\equiv \gamma^\mu_{\alpha\beta\gamma} + \delta\Gamma^\mu_{\alpha\beta\gamma}, \\ \gamma^\mu_{RAA} = \gamma^\mu_{AAR} &= \gamma^\mu, \quad \text{otherwise} \quad 0. \end{aligned} \tag{2·9}$$

where $\delta\Gamma^\mu_{\alpha\beta\gamma}$ denotes the HTL resummed contribution to the vertex function.[4]

As mentioned in the introduction, at zero temperature the fermion wave function renormalization constant $A(P)$ is equal to unity in the Landau gauge ($\xi = 0$) even in the ladder (point-vertex) DS equation, while at finite temperature it is not. The quantity $C(P)/A(P) \equiv M(P)$ plays the role of the mass function, in which we are interested, that vanishes in the chiral symmetric phase.

2.2. *The HTL resummed DS equations for the invariant functions A, B and C*

In the present analysis, we solve the DS equation for the retarded fermion self-energy function Σ_R, with the HTL resummed gauge boson propagator, by adopting further the following two approximations, i) the point-vertex approximation, and ii) the modified instantaneous exchange approximation, on which we give brief explanations below.

i) Point-vertex approximation

As for the vertex function $^*\Gamma^\mu$ we adopt the point-vertex approximation, namely we disregard $\delta\Gamma^\mu_{\alpha\beta\gamma}$ in Eq. (2·9). Thus we investigate the ladder (point-vertex) DS equation with the HTL resummed gauge boson propagator.

There are two reasons; Firstly, without the point-vertex approximation the numerical calculation we should carry out becomes so complicated that we can not manage with the power of the computer we use, because the HTL resummed contribution to the vertex function, $\delta\Gamma^\mu_{\alpha\beta\gamma}$, is the non-local interaction term, and also because it behaves singular in numerical calculations. Secondly, in the DS equation with the HTL resummed vertex function, it is difficult to resolve the problem of double counting of diagrams,[10] especially at the level of numerical analyses. Being free from this problem is the main reason we make use of the point-vertex approximation.

ii) Modified Instantaneous Exchange (MIE) approximation

We make use of the modified instantaneous exchange (IE) approximation (i.e., set the energy component of the gauge boson to be zero) to the gauge boson propagator, which consists of taking the IE limit in the HTL resummed longitudinal (electric) gauge boson propagator, ${}^*G_L^{\mu\nu}$, that is proportional to $B^{\mu\nu}$, while keeping the exact HTL resummed form for the transverse (magnetic) gauge boson propagator, ${}^*G_T^{\mu\nu}$, that is proportional to $A^{\mu\nu}$, and also for the massless gauge term in proportion to $D^{\mu\nu}$. The reason why we do not take the IE limit to the transverse mode is that the IE approximation reduces the transverse mode to the pure massless propagation, thus makes the important thermal effect, i.e., the dynamical screening of transverse propagation disappear. On the contrary, the longitudinal mode always receives the non-vanishing finite Debye screening effect, thus even if we take the IE limit the important thermal effect still survives in this mode.

With the above two approximations, we obtain the HTL resummed improved ladder DS equations for the invariant scalar functions A, B and C:

$$
\begin{aligned}
p^2[1-A(P)] = e^2 \int \frac{d^4K}{(2\pi)^4}\Bigg[& \{1+2n_B(p_0-k_0)\} Im[\,{}^*G_R^{\rho\sigma}(P-K)] \times \\
& \Big[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) - k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) + pkz g_{\sigma\rho} \\
& +2p_0 k_0 g_{\sigma 0} g_{\rho 0}\}\frac{A(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2} + \{P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0} \\
& -2p_0 g_{\sigma 0} g_{\rho 0}\}\frac{k_0+B(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\Big] + \{1-2n_F(k_0)\} \times \\
& {}^*G_R^{\rho\sigma}(P-K) Im\Big[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) - k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) \\
& +pkz g_{\sigma\rho} + 2p_0k_0 g_{\sigma 0} g_{\rho 0}\}\frac{A(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2} \\
& +\{P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0} - 2p_0 g_{\sigma 0} g_{\rho 0}\}\frac{k_0+B(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\Big]\Bigg] ,
\end{aligned} \tag{2·10}
$$

$$
\begin{aligned}
B(P) = e^2 \int \frac{d^4K}{(2\pi)^4}\Bigg[& \{1+2n_B(p_0-k_0)\} Im[\,{}^*G_R^{\rho\sigma}(P-K)] \times \\
& \Big[\{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} - 2k_0 g_{\sigma 0} g_{\rho 0}\}\frac{A(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2} \\
& +\{2g_{\rho 0}2g_{\sigma 0} - g_{\sigma\rho}\}\frac{k_0+B(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\Big] + \{1-2n_F(k_0)\} \times \\
& {}^*G_R^{\rho\sigma}(P-K) Im\Big[\frac{A(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} \\
& -2k_0 g_{\sigma 0} g_{\rho 0}\} + \frac{k_0+B(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\{2g_{\rho 0}2g_{\sigma 0} - g_{\sigma\rho}\}\Big]\Bigg] ,
\end{aligned} \tag{2·11}
$$

$$
\begin{aligned}
C(P) = -e^2 \int \frac{d^4K}{(2\pi)^4} & g_{\sigma\rho}\{1+2n_B(p_0-k_0)\} Im[\,{}^*G_R^{\rho\sigma}(P-K)] \times \\
& \Big[\frac{C(K)}{[k_0+B(K)+i\epsilon]^2 - A(K)^2k^2 - C(K)^2} + \{1-2n_F(k_0)\} \times
\end{aligned}
$$

$$ {}^{*}G_R^{\rho\sigma}(P-K)Im\Big[\frac{C(K)}{[k_0+B(K)+i\epsilon]^2-A(K)^2k^2-C(K)^2}\Big]\Big] \ , \tag{2·12} $$

The HTL resummed DS equations are the coupled integral equations for the six unknown functions because the invariants A, B and C have both real and imaginary parts. Therefore, the DS equations, Eqs. (2·10)-(2·12), are still quite tough to be solved even if we adopt the above two approximations.

2.3. *The effective potential $V[S_R]$ for the retarded full fermion propagator S_R*

The above DS equations, Eqs. (2·10)-(2·12), may have several solutions, and we choose the "true" solution by evaluating the effective potential $V[S_R]$ for the fermion propagator function S_R, then finding the lowest energy solution. The effective potential is expressed as[11)]

$$ \begin{aligned} V[S_R] =& i\mathrm{Tr}\,[\not{P}S_R] + i\mathrm{Tr}\ln\left[iS_R^{-1}\right] \\ &-\frac{e^2}{2}\int\frac{d^4K}{(2\pi)^4}\int\frac{d^4P}{(2\pi)^4}\frac{1}{2}\mathrm{tr}\,\big[\gamma_\mu S_R(K)\gamma_\nu S_R(P)D_C^{\mu\nu}(P-K) \\ &\qquad +\gamma_\mu S_C(K)\gamma_\nu S_R(P)D_R^{\mu\nu}(P-K)+\gamma_\mu S_R(K)\gamma_\nu S_C(P)D_A^{\mu\nu}(P-K)\big] \ , \end{aligned} \tag{2·13} $$

where the 1st and the 2nd terms correspond to the one-loop contribution, while the 3rd term corresponds to the two-loop contribution.

§3. Solution of the HTL improved ladder DS equation and its gauge-dependence

In this section we solve the HTL improved ladder DS equation derived in the previous section numerically by an iterative method. To start with we choose appropriate initial trial functions for $A(P)$, $B(P)$ and $C(P)$ with the guess for them to have non-trivial imaginary parts. By calculating the right-hand sides of Eqs. (2·10)-(2·12) we get "solutions" $A(P)$, $B(P)$ and $C(P)$, which are supposed to be better approximations to the real solutions. Replacing the initial trial functions for $A(P)$, $B(P)$ and $C(P)$ by the "solutions" thus got, we obtain further better approximations to the solutions. This procedure is iterated till we obtain converged solutions. Since the solutions thus determined may depend on the initial choice of the trial functions, we need to try various initial trial functions to find the real solutions. Some technical details in the numerical analysis in solving the DS equations are given in the Appendix.

If more than two converged solutions are obtained, among the solutions we adopt such a solution to be the true one that has the lowest value of the effective potential $V[S_R]$, Eq. (2·13).

To investigate how the solution thus obtained depends on the choice of gauges, we solve the HTL improved ladder DS equation by choosing various gauges within the covariant gauge. In the present paper, we carry out the analysis by choosing gauges mainly in the neighborhood of the Landau gauge ($\xi = 0$), and show explicitly the obtained solutions suffer from the strong gauge-dependence.

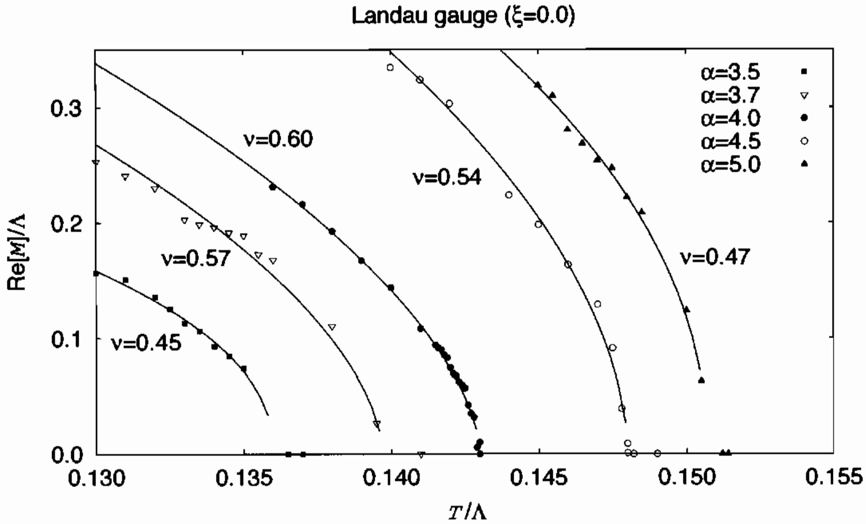

Fig. 1. The T-dependence of the mass $Re[M] = Re[C/A]$ at $p_0 = 0$, $p = 0.1\Lambda$ for various fixed values of the coupling constant α in the Landau gauge ($\xi = 0.0$). The best-fit curves at each coupling constant, with the critical exponents ν given in each data, are also shown.

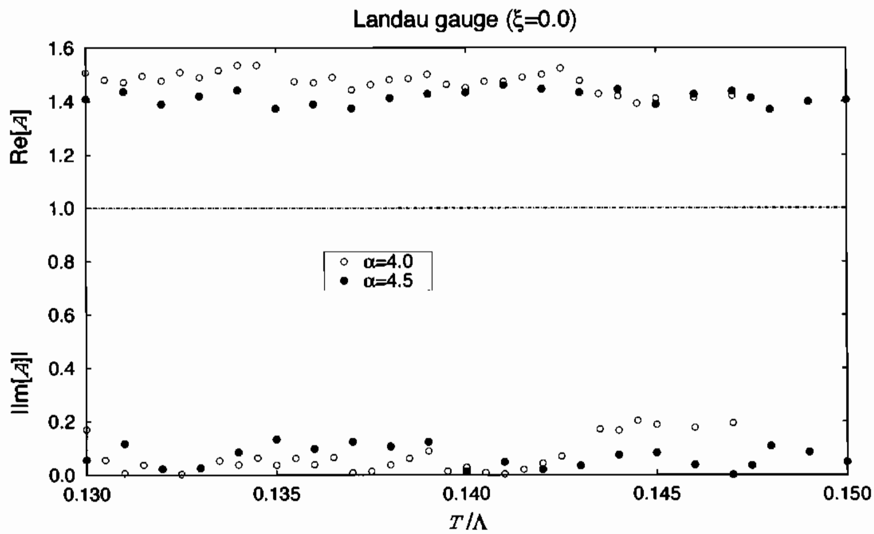

Fig. 2. Comparison of the wave function renormalization constant $Re[A]$ and $|Im[A]|$ at the coupling constant $\alpha = 4.0$ and 4.5 evaluated at $p_0 = 0.0$, $p = 0.1\Lambda$ in the Landau gauge ($\xi = 0.0$).

Gauges we choose are those with five different values of the gauge-parameter ξ, including the Landau gauge:

$\xi = 0$(Landau gauge), $\xi = \pm 0.05$ and $\xi = \pm 0.025$.

3.1. *Result in the Landau gauge* ($\xi = 0.0$)

Firstly we give the result in the Landau gauge, which is the gauge studied in most of the preceding analyses. Fig. 1 shows the mass $Re[M(P)] \equiv Re[C(P)/A(P)]$ as a function of temperature T at five different values of the coupling constant α, $\alpha = 3.5$, 3.7, 4.0, 4.5 and 5.0. In this figure we also give the critical exponent ν at each coupling constant α defined by

$$Re[M] = C_0 \, (T_c - T)^{\nu} \,, \quad T < T_c, \tag{3·1}$$

which controls how the mass $Re[M]$ vanishes near the critical temperature T_c. We see that the temperature dependence of the mass $Re[M]$ near the critical temperature T_c can be well described by the functional form Eq. (3·1) with a little dependence of the critical exponent on the couplng constant. The average value $\nu \simeq 0.527$ may also reproduce the results with appropriate C_0. Thus we may say that the phase transition takes place through the second order transition.*)

However, this result can not be justified in having a physical significance without further consideration. As shown in Fig. 2, the invariant function $A(P)$ at the momentum where the fermion mass is defined deviates largely from 1, even its imaginary part being sizable: $Re[A] \gtrsim 1.4$ and $|Im[A]| \simeq 0.1 \sim 0.2$, indicating Z_2 significantly smaller than 1, not even the real number. This fact implies that the result obtained in the Landau gauge apparently contradicts with $Z_1 = Z_2$ implied by the Ward-Takahashi identity, therefore we should not give a serious physical meaning to the result in the Landau gauge.

3.2. *Gauge-dependence of the solution*

Now we compare the results with the five different values of ξ in order to see how large the solution depends on the choice of gauge. We illustrate it by showing the results obtained in the case of the coupling constant $\alpha = 4.0$. Fig. 3 shows how the mass $M(P)$, as a function of temperature T, depends on the choice of gauge-parameters.

In this figure we give also the critical exponent ν at each value of the gauge-parameter. In any gauge the temperature-dependence of the mass $Re[M]$ near the critical temperature can be well described by the functional form Eq. (3·1) again with a little dependence of the critical exponent on the choice of gauge-parameter. The average value $\nu \simeq 0.527$ may also reproduce the results with appropriate C_ν. This fact indicates that the phase transition takes place through the second order transition.

However, as can be easily seen, the critical temperature strongly depends on the choice of gauge-parameter, demonstrating that it is essential to choose an appropriate gauge such that at least the identity $Z_1 = Z_2$ required by the Ward-Takahashi identity holds in solving the improved ladder DS equation in order to obtain reliable results which have predictive power.

*) To be exact, in order to conclude that the phase transition is of the second order, we must study at each coupling constant whether there are no stable states other than the one shown in the figure.

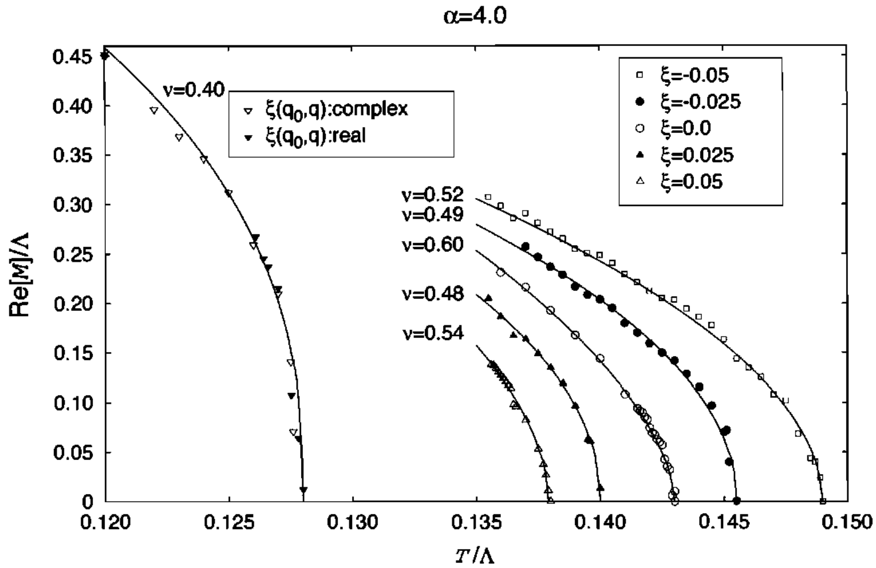

Fig. 3. Gauge-parameter-dependence of the fermion mass $Re[M] = Re[C/A]$ at the coupling constant $\alpha = 4.0$ evaluated at $p_0 = 0$, $p = 0.1\Lambda$. The best-fit curves for each value of the gauge-parameter, with the critical exponents ν given in each data, are also shown.

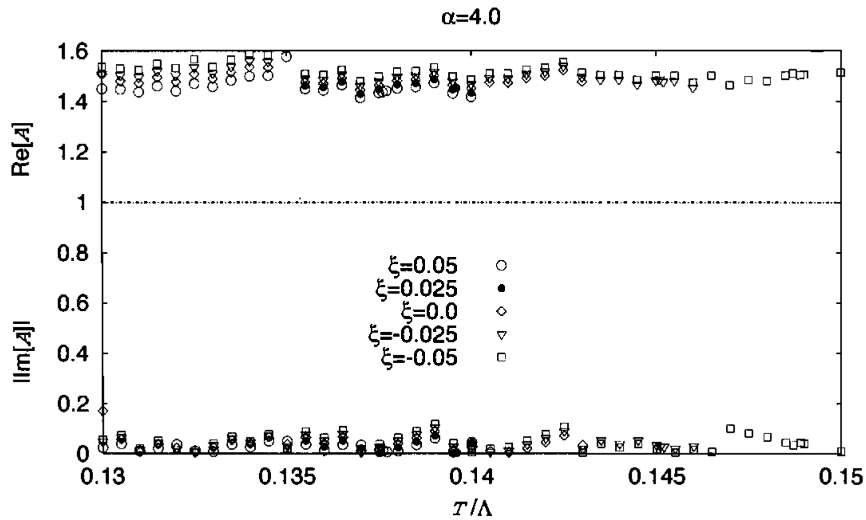

Fig. 4. Comparison of the wave function renormalization constant $Re[A]$ and $|Im[A]|$ at the coupling constant $\alpha = 4.0$ evaluated at $p_0 = 0$, $p = 0.1\Lambda$.

For further consideration of the optimal gauge, we show the gauge dependence of the wave-function renormalization function $A(P)$ in Fig. 4. Fig. 4 shows the real and imaginary parts of $A(P)$ as a function of temperature for the five gauge parameters. As can be seen from this figure the Landau gauge does not give any better property compared with other gauge parameters: obviously the Ward-Takahashi identity does not favor the Landau gauge at all !

3.3. *Further analysis on the gauge-dependence of the solution*

Above results given in 3.2. may meet with following criticism; In the present framework of analysis the cutoff parameter Λ used to scale dimensionful quantities is not a physical quantity, and so we should rather use the critical temperature T_c as a scale-dimension, then the observed gauge-parameter dependence might disappear.

To answer the above criticism we carry out the renormalization (or, re-scaling) of Λ so that the critical temperature T_c always coincides irrespective of the choice of gauges. By using in each gauge the renormalized gauge-dependent cutoff parameter $\Lambda(\xi)$ to scale dimensionful quantities, we study whether or not the observed gauge-dependence can be absorbed into such a *gauge-dependent* renormalization. Since the cutoff parameter Λ is not a physical quantity, it is surely possible to perform such a renormalization of Λ.

The reason why we adopt the prescription to renormalize (or, re-scale) Λ gauge-by-gauge, not taking the prescription to use the critical temperature T_c as a scale-dimension, is the following; By performing the renormalization gauge-by-gauge, we can clearly see whether *a single renormalization-prescription in a given gauge* can absorb at once all the gauge-dependences of the mass function at different strength of the coupling constant, or not. If it is the case, then we can say that the gauge-dependence of mass function is absorbed into the gauge-dependent renormalization-prescription and the mass function looks gauge-independent at least around the Landau gauge. If not, however, we must choose a different renormalization-prescription for each different strength of the coupling constant as well as for each different choice of gauges, thus we can say gauge-dependence of the mass function cannot be made to disappear through any admissible prescription.

As a start, first we study the data for the mass function at $\alpha = 4$, given in Fig. 3. For convenience, here we perform the renormalization (or, re-scaling) of Λ gauge-by-gauge so that the critical temperature T_c coincides with that of the Landau gauge, and define the finite renormalization (or, re-scaling) factor $z(\xi)$ by

$$z(\xi) = \Lambda/\Lambda(\xi) = T_c(\xi = 0)/T_c(\xi), \qquad (3\cdot2)$$

where $T_c(\xi)$ is the gauge-dependent critical temperature determined in Fig. 3, and $\Lambda(\xi)$ is the renormalized gauge-dependent cutoff parameter used to scale all dimensionful quantities, e.g., the mass function M, the temperature T, the three-momentum p and the energy p_0, in the corresponding gauge. $T_c(\xi = 0)$ is the critical temperature in the Landau gauge determined in the scale of Λ, the original cutoff parameter, in Fig. 3.

If all the mass functions in different gauges lie on a single curve after this renormalization and re-scaling, then we can conclude that the gauge-dependece observed in Fig. 3 are absorbed into the *gauge-depnedent renormalization-prescription* at least at the value of the coupling constant $\alpha = 4$.

Result of the analysis is shown in Fig. 5, a)$\alpha = 4$. In Fig. 5, the horizontal/vertical line represents the temperature/mass function in the scale of $\Lambda(\xi) =$

$z^{-1}(\xi)\Lambda$ and the mass function is evaluated at $p_0 = 0$ and $p = 0.1$ in the scale of $\Lambda(\xi)$. As can be seen *the fermion mass functions in different gauges lie on different curves* as a function of temperature. Thus Fig. 5 a) shows that there remains a clear gauge-dependence in the behavior of the mass function, namely, the fermion mass in the medium takes different values depending on the choice of gauges. Here it is worth noticing that the critical exponent does not change with the the present renormalization (or, re-scaling) of Λ.

Next let us study what happens to the data for the mass function at different strength of the coupling constant $\alpha = 4.5$, with the same gauge-by-gauge renormalization- (or, re-scaling-)prescription carried out at $\alpha = 4$, namely, with the same value of the finite renormalization factor $z(\xi)$ determined in the above analysis at $\alpha = 4$. All dimensionful quantities in each gauge are re-scaled by this value of the finite renormalization factor $z(\xi)$ at $\alpha = 4$.

We give the result in the same Fig. 5, b)$\alpha = 4.5$, showing that the critical temperatures in different gauges cannot coincide, indicating the large gauge-dependence of the critical temperature. The critical exponent also depends on the choice of gauges.

It is surely possible to choose a different renormalization-prescription such that the large gauge-dependence of the critical temperature can be absorbed into the distinctive renormalization of the cutoff parameter at each different strength of the coupling constant as well as in each different choice of gauges. However, we cannot find any admissible reason to adopt such a renormalization-prescription.

We can carry out the same analysis starting from the data for the mass function at $\alpha = 4.5$. Results are given in Fig. 6. Here we performed the the gauge-dependent renormalization (or, re-scaling) of Λ so that the critical temperature T_c coincides with that of the Landau gauge at $\alpha = 4.5$, and all dimensionful quantities in each gauge are re-scaled by this gauge-dependent renormalized cutoff parameter $\Lambda(\xi)$ determined at $\alpha = 4.5$. The mass functions $Re[M]/\Lambda(\xi)$ at $\alpha = 4.5$ in various gauges show almost the same pattern of behavior as those at $\alpha = 4$, Fig. 5 a), as functions of temperature, see Fig.6 a)$\alpha = 4.5$. With this gauge-dependent renormalized cutoff parameter $\Lambda(\xi)$, namely, with the same value of the finite renormalization factor $z(\xi)$ determined at $\alpha = 4.5$, the data for the mass function at $\alpha = 4$, given in Fig. 6 b)$\alpha = 4$, again shows the large gauge-dependence: the critical temperatures in different gauges cannot coincide and the critical exponent also depends on the choice of gauges.

We can also perform the analysis of mass function at fixed temperature (in the scale of renormalized cutoff $\Lambda(\xi) = z^{-1}(\xi)\Lambda$) as a function of the coupling constant α. In Fig. 7 shown are results of analyses at two temperatures, $T/\Lambda(\xi) = 0.110$ and 0.143, the latter being the critical temperature $T_c(\xi)/\Lambda(\xi)$ at $\alpha = 4$ (see, Fig. 5 a)). Here we also use the same value of the finite renormalization factor $z(\xi)$ determined in the above analysis at $\alpha = 4$.

As can be seen, the mass functions $Re[M]/\Lambda(\xi)$ at $T/\Lambda(\xi) = 0.143$ coincide at $\alpha = 4$, and those at $T/\Lambda(\xi) = 0.110$ also show, though not so clear, the same tendency as α approches to 4. These behaviors arise due to the choice of present renormalizaton-prescription, i.e., the choice of $\Lambda(\xi)$ determined at $\alpha = 4$. However,

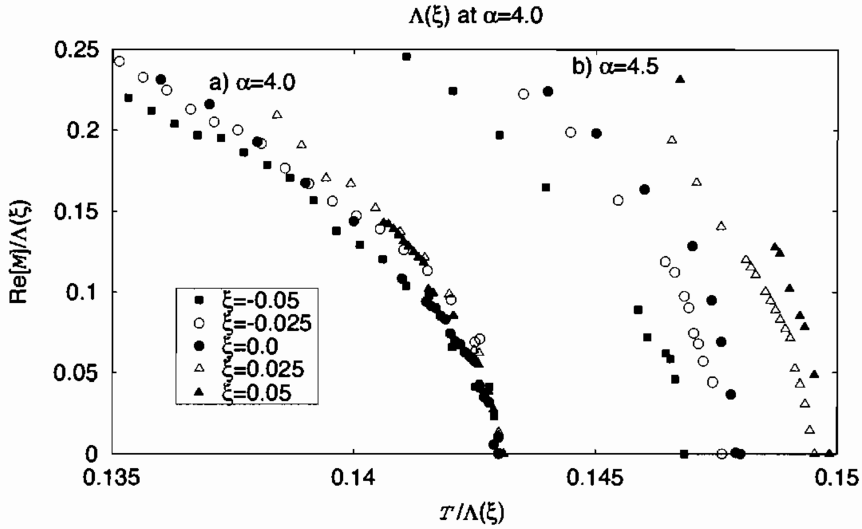

Fig. 5. Fermion mass $Re[M]/\Lambda(\xi)$ as a function of "re-scaled" temperature $T/\Lambda(\xi)$ at a) $\alpha = 4.0$ and b) $\alpha = 4.5$, evaluated at $p_0 = 0$ and $p = 0.1\Lambda(\xi)$. The renormalized cutoff (or, scale) parameter $\Lambda(\xi)$ is determined gauge-by-gauge by Eq. (3·2) at $\alpha = 4.0$ (see, text).

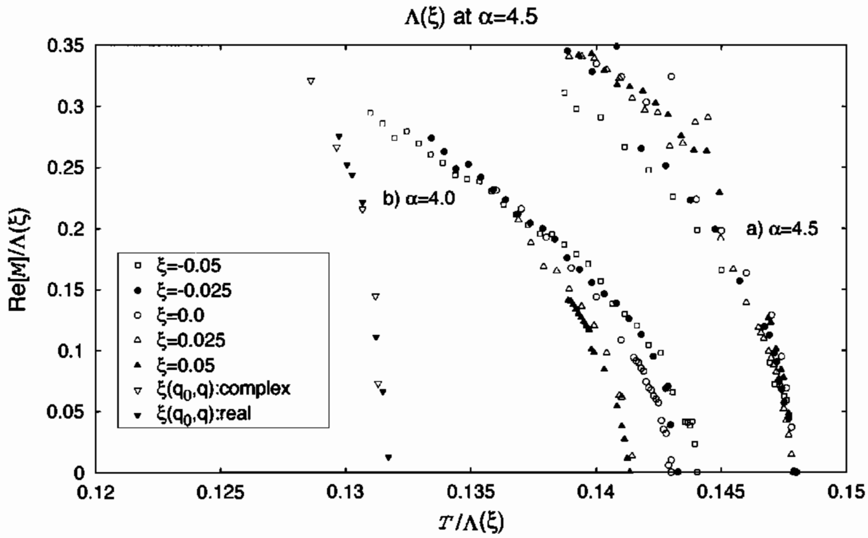

Fig. 6. Fermion mass $Re[M]/\Lambda(\xi)$ as a function of "re-scaled" temperature $T/\Lambda(\xi)$ at a) $\alpha = 4.5$ and b) $\alpha = 4.0$, evaluated at $p_0 = 0$ and $p = 0.1\Lambda(\xi)$. The renormalized cutoff (or, scale) parameter $\Lambda(\xi)$ is determined gauge-by-gauge by Eq. (3·2) at $\alpha = 4.5$ (see, text).

as the coupling constant α departs from $\alpha = 4$, the mass function becomes separate: the larger α departs from $\alpha = 4$, the bigger the separation of the mass function becomes.

From the above analyses we must conclude that the observed gauge-dependence of the fermion mass function cannot be absorbed into a gauge-dependent renormal-

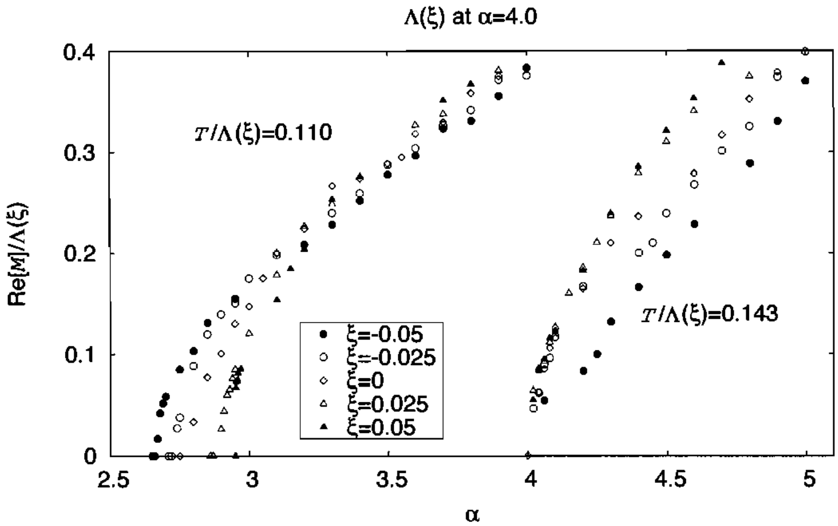

Fig. 7. Fermion mass $Re[M]/\Lambda(\xi)$ as a function of coupling constant α at two fixed "re-scaled" temperature $T/\Lambda(\xi) = 0.143$ and 0.110, evaluated at $p_0 = 0$ and $p = 0.1\Lambda(\xi)$. The renormalized cutoff (or, scale) parameter $\Lambda(\xi)$ is determined gauge-by-gauge by Eq. (3·2) at $\alpha = 4.0$ (see, text).

ization of the cutoff parameter Λ. The fermion mass function determined by solving the improved ladder DS equation really depends on the choice of gauges, so do in general the critical temperature T_c, the critical coupling α_c and the critical exponents. The only physical conclusion we can get from the analysis of the improved ladder DS equation in the general covariant gauge is that the chiral phase transition in massless thermal QED/QCD takes place through the second order transition.

Thus we should finally say that it is essential to choose an appropriate gauge such that the identity $Z_1 = Z_2$ implied by the Ward-Takahashi identity holds in solving the improved ladder DS equation in order to obtain physically meaningful results, having predictive power.

§4. Conclusions and discussion

In the present paper we solved with a numerical procedure the HTL improved ladder DS equation for the retarded fermion self-energy function Σ_R to study the spontaneous generation of fermion mass in thermal QCD/QED, mainly focussing on the gauge-dependence of the solution within a general covariant gauge where the gauge parameter ξ is any constant number. It should be noticed that, in the DS equation in the point-vertex ladder approximation, no solution receives the vertex correction, thus the vertex renormalization constant Z_1 is exactly unity, $Z_1 = 1$. We also made an improvement in estimating the numerical integration over singular parts of the integration kernel in the present analysis.

With the numerical solutions thus obtained in various gauges, we found the followings;

i) In the Landau gauge the obtained solution shows a significant change compared to the simple ladder analyses,[3] indicating the importance of taking the dominant effect of thermal fluctuations into the integration kernel through the HTL resummation procedure.
ii) In any gauge (including the Landau gauge $\xi = 0$) where the gauge parameter ξ being any constant number, the fermion wave function renormalization function $A(P)$ always deviates largely from unity even at the momentum where the mass is defined. This fact clearly shows that the corresponding solutions explicitly contradict with the identity $Z_1 = Z_2$ implied by Ward-Takahashi identity, which makes the physical meaning of the solution being obscure.
iii) The obtained solutions strongly depend on the choice of gauge parameters: the critical temperatures (and the critical coupling constants) change significantly gauge by gauge.
iv) Strong gauge-dependence of the mass function was further confirmed; We performed the renormalization of the cutoff parameter Λ gauge-by-gauge and made the re-scaling, by this gauge-dependent cutoff parameter $\Lambda(\xi)$, of all dimensionful quantities, e.g., the mass function M, the temperature T, the three-momentum p and the energy p_0 in the corresponding gauge, then studied whether or not the observed gauge-dependence could be absorbed into such a gauge-dependent renormalization. The result was negative: strong gauge-dependences of the mass function $M/\Lambda(\xi)$ still survive, showing the fermion mass in the medium takes different values depending on the choice of gauges.
v) We also determined the critical exponent ν defined by Eq. (3·1), which controls how the mass $Re[M]$ vanishes near the critical temperature T_c. The results show that the temperature-dependence of mass $Re[M]$ near the critical temperature T_c can be well described by the functional form Eq. (3·1), and that ν also depends on the strength of the coupling, and on the choice of gauge. Thus the only conclusion we could get from the analysis of the improved ladder DS equation in the general covariant gauge is that the chiral phase transition in massless thermal QED/QCD takes place through the second order transition.

All the above findings show the solution of the HTL improved ladder DS equation suffers from the problem of large gauge-dependence within a general covariant gauge where the gauge parameter ξ is any constant number. Namely the solution varies significantly gauge by gauge. The most serious problem we face is there is no definite criterion which solution we should choose, which then reminds us of the fact that at zero temperature any solution in the Landau gauge of the DS equation with the ladder kernel automatically satisfies $Z_1 = Z_2$ implied by the Ward-Takahashi identity: one of the most promising criterion selecting the solution to have definite physical meaning.

Here we give some comment on the choice of the gauge in the present analysis. As we noted in §3, we solved the HTL improved ladder DS equation by choosing gauges only in the neighborhood of the Landau gauge ($\xi = 0$). The reason why we choose such gauges is as follows; i) The Landau gauge has a special significance at zero temperature, and might do so even at finite temperatures. With this expectation many analyses have been carried out in the Landau gauge, thus in performing the analysis in the neighborhood of the Landau gauge, we can see what really happens by comparing the result of our analysis with those of the preceding works. ii) There is a more practical reason: in our present procedure we can get nicely converged numerical solutions mainly in the gauges neighboring with the Landau gauge.

Anyway with such a small change of gauge, the solution we obtained shows a big change in the "physical quantities" such as the critical temperature.

Thus the only conclusion we could have from our analysis is that the chiral phase transition in massless thermal QED/QCD at zero fermion number density takes place through the second order transition. To determine the critical temperature, the critical coupling constant and also the corresponding critical exponents in a physically sensible way, we should find such a solution that satisfies at least $Z_1 = Z_2$, required by the Ward-Takahashi identity.

Needless to say, any solution of the DS equation in the ladder approximation can not satisfy the full Ward-Takahashi identity, stating the identity between the vertex function and the derivative of the fermion self-energy function. We only propose a possible choice of gauge where $Z_1 = Z_2$ holds at least, which may help us to get a physically meaningful solution at the same level of significance as that at zero temperature analysis in the Landau gauge. It is our next plan of analysis to carry out this procedure and investigate the properties of such solutions.[12),13)]

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Appendix A

—— *Details of the numerical analysis of the DS equation* ——

In this Appendix, firstly we explain the problems in the numerical analysis, which we face in solving numerically the DS equations, Eqs. (2·10)-(2·12), for the invariant functions A, B and C, then give the procedures we make use of to resolve the problems. We face two problems in the numerical analysis;

i) As can be seen from Eq. (2·10), to determine the function A, firstly we perform the integration over the variable K in the right-hand-side (r.h.s.) of the equation, then divide the result by p^2. Needless to say, the p-dependences of both sides of the equation should agree with each other, and as it is easy to confirm that no problem appears in the analytical calculation. However, a problem does appear in performing the numerical calculation. In the small-p region, we are forced to carry out the numerical integration in the r.h.s. in a higher accuracy level compared with other region of the momentum p. This is in fact a hard task, causing larger errors and thus being the origin of the unstable behavior in the numerically determined A in the small-p region. This problem becomes especially serious in the contribution coming from the term that depends explicitly on the gauge-parameter ξ, i.e., the $D^{\mu\nu}$ term in the gauge boson propagator

${}^*G^{\mu\nu}$, Eq. (2·2). Formally we must divide by p^3, not by p^2, to determine this contribution to the function A.

ii) There are several singular terms in the gauge boson propagator ${}^*G^{\mu\nu}$, Eq. (2·2), appearing in the integration kernel of the DS equations; a) The ξ-dependent $D^{\mu\nu}$ term is a pure massless double-pole mode. b) The transverse (magnetic) mode being proportional to $B^{\mu\nu}$ receives the so-called dynamical screening, but it becomes massless when the energy-component of the momentum vanishes with the space-component of it being finite.

In both cases we face the numerical integration over singular functions, e.g., the principal part and the δ-function. In the analytical calculation these functions do not cause any trouble, but in the numerical calculation they, especially the principal part, do cause troubles to obtain stable solutions. The procedures we make use of in order to resolve the above problems are as follows;

i) As for the first problem i) above, we carry out the integration in the r.h.s. of Eq. (2·10) by two different methods depending on the region of the momentum p. In the small-p region $p < p_{th}$, as for the contribution coming from the $D^{\mu\nu}$ term that depends explicitly on the gauge-parameter ξ, we expand the corresponding integrand of the r.h.s. of Eq. (2·10) in the power series of p, keeping up to the p^3 term, then carry out the integration. In the large-p region $p > p_{th}$, we perform the ordinary numerical integration to get the function A. The explicit value of the "threshold momentum" p_{th} is determined by considering the stability as well as the smoothness of the solution. In the present analysis we choose $p_{th} = 0.2\Lambda$.

ii) As for the second problem, we use the ordinary procedure. When the integration over the principal part appears, we divide the integration region into two parts: the integration in the neighborhood of the singular point, and the integration away from the singular point. We can carry out the simple numerical integration away from the singular point. The integration in the neighborhood of the singular point is carried out analytically, by taking the unknown functions A, B, and C kept constants with the values of those at the singular point of the gauge boson propagator.

In order for this method to work, the unknown functions A, B and C should behave smooth in the neighborhood of the singular point of the gauge boson propagator, and also we should take the neighborhood of the singular point as narrow as possible. The first point can be checked a posteriori by the obtained solutions, and the result is satisfactory. As for the size of the neighborhood where the integration is performed analytically, we must choose by seeing the stability and the smoothness of the obtained solutions. In the present analysis, we choose $|p_0 - k_0| < 0.04\Lambda$ and $|\mathbf{p} - \mathbf{k}| < 0.2\Lambda$.

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