Current Algebra Sum Rules for Inclusive Pion Productions

— Finite Energy Sum Rule based on the Current Algebra —

Hisao NAKKAGAWA

Institute for Natural Science, Nara University

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Current algebra sum rule, which relates the intergral of the inclusive cross section of pions to the total cross section, is derived. It is shown that this sum rule presents a new type of finite energy sum rule in order to calculate the pomeron component.

With the hypotheses of the current algebra (CA) and the partially conserved axial-vector current (PCAC), we have succeeded to derive many beautiful sum rules, and to extract out fruitful ideas.[1] Almost all of the existing sum rules are, however, obtained by sandwiching the current-commutator with stable one-particle states, and have the form relating the integral of the total cross section to some constant. Another type of sum rules starting from the current-*anti*commutator can also be obtained.[2] The derivation of these sum rules relies on the Deser-Gilbert-Sudarshan representation[3] of a stable one-particle matrix element. Namely such sum rules have been derived by sandwiching the current-anticommutator again with stable one-particle states, thus have the same structure as the above-mentioned CA sum rules.

During the 1970's semi-inclusive reactions initiated with both hadron-hadron and lepton-hadron are widely measured, and much has been studied by using the inclusive sum rules and various dynamical models.[4] However, few attempts to use the hypotheses of CA and PCAC to these reactions are there.[5] Some pioneering works in order to derive sum rules for the semi-inclusive cross sections by sandwiching the current-commutator with two-particle scattering states have been done,[6] but meaningful sum rule could not be obtained.

The purpose of the present paper is to show that, despite of the negative result until now, we can in fact get the CA sum rules for the semi-inclusive cross sections. With the use of a method slightly different from the one in calculating the one-particle matrix elements, we give simple derivation of the CA sum rules for inclusive pion production, and also study its indications.

We assume the following:

i) The light-like charges

$$\hat{Q}=\int dx_- d^2x_\perp \mathcal{J}^+ (\equiv \int d^3x_- \mathcal{J}^+(x)) \tag{1}$$

satisfy the algebra

$$[\hat{Q}_+^5, \hat{Q}_-^5]=2\hat{Q}_3, \tag{2}$$

where $\hat{Q}_\pm^5 \equiv (\hat{Q}_1^5 \pm i\hat{Q}_2^5)/\sqrt{2}$.

ii) Divergence of an axial-vector current is proportional to the pion field

$$\partial_\mu \mathcal{J}_a^{5\mu}(x)=c\phi_a(x), \quad c=\sqrt{2}M_N M_\pi^2 g_A/g_r(0). \tag{3}$$

We also use the final state density operator[7]

$$\rho=T|p_1p_2{}^{out}><p_1'p_2'{}^{out}|T^+ \tag{4}$$

with the trace property

$$\begin{aligned}\mathrm{Tr}\{\rho\} &\equiv \sum_n <n^{out}|T|p_1p_2{}^{out}><p_1'p_2'{}^{out}|T^+|n^{out}> \\ &=2\sqrt{\Delta(s)}\sigma_{tot}(s)(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2'),\end{aligned} \tag{5}$$

where T is the scattering matrix defined as $S=1+iT$, and $\Delta^{1/2}(s)$ is the usual flux factor, i. e.,

$$\Delta(s)=[s-(m_1+m_2)^2][s-(m_1-m_2)^2]. \tag{6}$$

Now consider the operator

$$2\hat{Q}_3\rho=[\hat{Q}_+^5, \hat{Q}_-^5]\rho \tag{7}$$

and take the trace of both sides:

$$\begin{aligned}&2\sum_n <n^{out}|\hat{Q}_3T|p_1p_2{}^{out}><p_1'p_2'{}^{out}|T^+|n^{out}> \\ &=\sum_n <n^{out}|\hat{Q}_-^5T|p_1p_2{}^{out}><p_1'p_2'{}^{out}|T^+\hat{Q}_+^5|n^{out}>-(+\leftrightarrow -).\end{aligned} \tag{8}$$

In contrast to the ordinary derivation of the CA sum rules, we now must consider the matrix elements of the commutator, Eq. (2), between two-particle scattering states, i. e., Eq. (8), and this brings in some complications.[6] The problems are i) the prescription how to extract out the completely connected contributions in both sides of Eq. (8), and ii) the estimation of the contribution from the three-particle process. The first problem is rather easily solved for the R. H. S. of Eq. (8), but should be carefully examined for the L. H. S.. The second problem is overcomed with the assumption that the completely connected part of the three-particle process should exibit ordinary Regge behavior.

I) Calculation of the R. H. S. of Eq. (8)

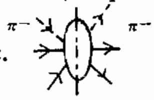
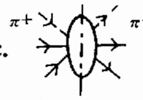
Carefully extracting out the completely connected parts from the matrix elements $<n^{out}|\hat{Q}_-^5T|p_1p_2{}^{out}>$, we have (by using Eq. (3)),

$$\begin{aligned}&\sum_n <n^{out}|\hat{Q}_-^5|p_1p_2{}^{in}>_c<p_1'p_2'{}^{in}|\hat{Q}_+^5|n^{out}>_c-(+\leftrightarrow -) \\ &=\sum_n \frac{4P_i^+P_i'^+}{(M_n^2-s)^2}\frac{c^2}{M_\pi^4}<p_1'p_2'{}^{in}|\int d^3x_- j_{\pi^+}(x)|n^{out}>_c \\ &\quad\times <n^{out}|\int d^3x_- j_{\pi^-}(x)|p_1p_2{}^{in}>_c-(+\leftrightarrow -) \\ &=(2\pi)^3\delta_+^3(P_i-P_i')2P_i'^+\frac{c^2}{\pi M_\pi^4}\int P_i^+dq_- \\ &\quad\times\sum_n\frac{1}{(M_n^2-s)^2}<p_1'p_2'{}^{in}|j_{\pi^+}(0)|n^{out}>_c<n^{out}|j_{\pi^-}(0)|p_1p_2{}^{in}>_c(2\pi)^4\delta^4(P_n+q-P_i) \\ &\quad-(+\leftrightarrow -)\end{aligned} \tag{9}$$

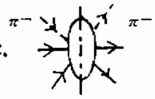
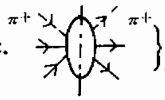
for the R. H. S. of Eq. (8), where $\delta_+^3(p)\equiv\delta(p_+)\delta^2(p_\perp)$, $q=(q_+=0, q_-, q_\perp=0)$, $P_i=$

p_1+p_2 and $P_i'=p_1'+p_2'$. Because we extract out only completely connected parts, Eq. (9) becomes, after setting $p_i=p_i'$ $(i=1,2)$,

$$(2\pi)^3\delta_+^3(0)2P_i^+\frac{c^2}{2\pi M_\pi^4}\left[\!\left[\sqrt{\Delta(s)}\int_{\nu_{th}}^{\nu_{max}}\frac{d\nu}{\nu^2}\left\{\frac{d\sigma^{\pi+}}{dq}-\frac{d\sigma^{\pi-}}{dq}\right\}\right|_{q_\perp=0}$$

$$+\int_{\nu_{max}}^{\infty}\frac{d\nu}{\nu^2}\left\{\text{Disc.}\right.$$  $$-\text{Disc.}$$  $$\left.\right\}\right]\!\right]$$

$$=(2\pi)^3\delta_+^3(0)2P_i^+\frac{c^2}{\pi M_\pi^4}\left[\!\left[\sqrt{\Delta(s)}\int_{W^2_{th}}^{W^2_{max}}\frac{dW^2}{(W^2-s)^2}\left\{\frac{d\sigma^{\pi+}}{dq}-\frac{d\sigma^{\pi-}}{dq}\right\}\right|_{q_\perp=0}$$

$$+\int_{W^2_{min}}^{\infty}\frac{dW^2}{(W^2-s)^2}\left\{\text{Disc.}\right.$$  $$-\text{Disc.}$$  $$\left.\right\}\right]\!\right], \qquad (10)$$

where $d\sigma^c/dq$ denotes the invariant cross section detecting a particle c, and $dq\equiv d^3q/(2\pi)^3 2q_0$. Variables appearing in Eq. (10) are the following:

$$\nu=q\cdot P_i=(s-W^2)/2,\quad W^2=M_n^2,$$
$$\nu_{max}=M_\pi\sqrt{s},\quad W_{max}=\sqrt{s}-M_\pi,\quad W_{min}=\sqrt{s}+M_\pi \qquad (11)$$

II) Calculation of the L. H. S. of Eq. (8)

$$2\sum_n\langle n^{out}|\hat{Q}_3T|p_1p_2^{out}\rangle\langle p_1'p_2'^{out}|T^+|n^{out}\rangle$$
$$=2i\langle p_1'p_2'^{out}|(T^+-T)\hat{Q}_3|p_1p_2^{out}\rangle. \qquad (12)$$

Using the Fourier transform of the local current $V_3^\mu(x)$, $\tilde{V}_3^\mu(q)$, we can calculate the matrix elements in Eq. (12) by the use of the low energy theorem of Low[8),6)].

For the sake of simplicity we consider the case of unpolarized proton-proton states. Generalization is easy and gives the same result (Eq. (18), below). Then, we have the following;

$$\langle p_1'p_2'^{out}|\tilde{V}_3^+(q)|p_1p_2^{in}\rangle$$
$$=(2\pi)^3\delta_+^3(P_i-P_i')\left[\frac{1}{2}(p_1+p_2+p_1'+p_2')^+F_1+\frac{1}{2}(p_1-p_2+p_1'-p_2')^+F_2+\cdots\right], \qquad (13)$$

where the rest parts in [......] vanishes in the limit of forward scattering, and

$$F_1=\frac{1}{2}I_3^p\left(\frac{1}{p_1\cdot q}-\frac{1}{p_1'\cdot q}+\frac{1}{p_2\cdot q}-\frac{1}{p_2'\cdot q}\right)A$$
$$+\frac{1}{2}I_3^p\left(4-\frac{p_2\cdot q}{p_1\cdot q}-\frac{p_2'\cdot q}{p_1'\cdot q}-\frac{p_1\cdot q}{p_2\cdot q}-\frac{p_1'\cdot q}{p_2'\cdot q}\right)\frac{\partial A}{\partial\nu}, \qquad (14)$$

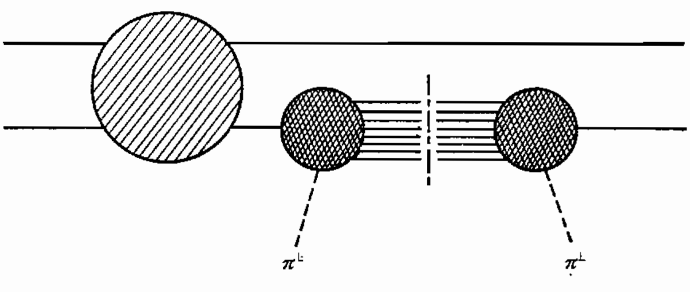


Fig. 1 Semi-connected diagrams.

A is the scattering amplitude for $pp\to pp$ and I_3^p is the third component of the proton isospin. Notice that the expression (14) contains the contributions from the semi-connected diagrams (see, Fig. 1). If we subtract out these contributions following the prescription given in the Appendix of Ref. 6, the remaining contributions corresponding to the completely connected parts — which are the one we are actually considering — become (by setting $p_i=p_i'$),

$$2i<p_1p_2^{out}|(T^+-T)\hat{Q}_3|p_1p_2^{out}>=\lim_{q\to 0}2i<p_1p_2^{out}|(T^+-T)\tilde{V}_3^+(q)|p_1p_2^{out}>$$

$$=(2\pi)^3\delta_+^3(0)2P_i^+4I_3\frac{\partial}{\partial s}(\mathrm{Im}\,A) \qquad (15)$$

where I_3 is the third component of the isospin of the initial state, and

$$\mathrm{Im}\,A=\sqrt{\Delta(s)}\sigma_{tot}(s). \qquad (16)$$

Now we take the limit $s\to\infty$. Remembering the fact that the pions being considered are the light-like ones, i. e., $q^2=0$, $q_+=0$ and $q_\perp=0$, the three-particle scattering amplitude may be replaced with the one in which pions are near their threshold, namely the one in which pions are treated by the external line insertions. Then the contribution from the three-particle processes to Eq. (10) can be estimated, by using the assumption that the three-particle amplitude shows the ordinary Regge behavior, as

$$\int_{W^2_{min}}^{\infty}\frac{dW^2}{(W^2-s)^2}\frac{2(W^2-s)}{W}\sim\frac{1}{\sqrt{s}}\ln\frac{2\sqrt{s}}{M_\pi}\underset{s\to\infty}{\longrightarrow}0. \qquad (17)$$

With this estimation, at sufficient high energies we have the following sum rule:

$$2I_3\frac{d}{ds}(\sqrt{\Delta(s)}\sigma_{tot}(s))$$

$$=\frac{M_N^2g_A^2}{\pi g_r^2(0)}\int_{W^2_{th}}^{\infty}\frac{\sqrt{\Delta(s)}dW^2}{(W^2-s)^2}\left\{\frac{d\sigma^{\pi+}}{dq}-\frac{d\sigma^{\pi-}}{dq}\right\}\Bigg|_{q_\perp=0} \qquad \text{(18-a)}$$

$$=\frac{16\pi M_N^2g_A^2}{g_r^2(0)}\int_{2M_\pi/\sqrt{s}}^{1}\frac{dx}{x}\sqrt{1+\frac{4M_\pi^2}{x^2s}}\left\{\frac{d^2\sigma^{\pi+}}{dxdq_\perp{}^2}-\frac{d^2\sigma^{\pi-}}{dxdq_\perp{}^2}\right\}\Bigg|_{q_\perp{}^2=0} \qquad \text{(18-b)}$$

$$\simeq\frac{16\pi M_N^2g_A^2}{g_r^2(0)}\int_0^1\frac{dx}{x}\left\{\frac{d^2\sigma^{\pi+}}{dxdq_\perp{}^2}-\frac{d^2\sigma^{\pi-}}{dxdq_\perp{}^2}\right\}\Bigg|_{q_\perp{}^2=0} \qquad \text{(18-c)}$$

To obtain Eq. (18-b, c), we assume the scaling behavior of the inclusive pion production cross-sections. As was noted earlier, this sum rule is independent of the initial states. In addition, it has a form closely connected to the Adler-Weisberger sum rule.

Here we briefly comment on the implications of this sum rule. Firstly, because we use the PCAC, the resulting sum rule may not reproduce the leading particle effects which are clearly seen in the experiments $\pi^\pm p\to\pi^\pm X$.[4] Therefore we expect that the non-leading particle component (NLPC) of the detected pions should satisfy this sum rule. Secondly, if the NLPC really scales at sufficiently high energies the behavior of the total cross section in such energy regions is governed by the small x behavior of $d^2\sigma(\pi^+-\pi^-)/dxdq^2$ near $q_\perp{}^2=0$ as follows:

$$\sigma_{tot}(s)\underset{s\to\infty}{\sim}\begin{cases}(\ln s)^r\\ s^{\alpha/2}\end{cases}\quad\text{for}\quad\frac{d^2\sigma(\pi^+-\pi^-)}{dxdq_\perp{}^2}\Bigg|_{q_\perp{}^2\simeq 0}\underset{x\sim 0}{\sim}\begin{cases}\left(\ln\frac{1}{x}\right)^{r+1}\\ x^{-\alpha}(\alpha\geq 0)\end{cases}$$

The case where the scaling violation exists is not considered here because the experimental situation whether there really exists the scaling violation in this NLPC or not is still confusing, and also because, even if there exist small scale-violating phenomena, their effects may not be significant in the above estimations.

Finally we make notice the following fact. The sum rule, Eq. (18), may be considered as a type of finite energy sum rule (FESR), relating the integral of the "non-pomeron" components of the three-particle to three-particle amplitude from the threshold to some sufficiently high energies, to the "pomeron" component of two-particle amplitude at that energies (with the neglection of terms of order $1/\sqrt{s}$). It seems to be an interesting question to be studied what emerges from this new FESR. More complete analysis of this sum rule and its implications, and the local current version of it will be given elsewhere.

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