Improving the Effective Potential at Finite Temperature*

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ABSTRACT

We propose a *simple* and *effective* procedure to improve the finite temperature effective potential so as to satisfy the renormalization group equations. We also demonstrate this procedure by explicit calculations at zero tempreraturе renormalization scheme.

I. Introduction

The effective potential (EP) at finite temperature is a convenient tool to investigate the phase transition of the relativistic quantum field theory[1)]. One of the common procedures for computing the EP is the perturbative calculation with the loop expansion[1)]. However, the perturbatively calculated L-loop approximation of the EP, $V^{(L)}$, suffers from the famous problem of renormalization-scheme (RS) dependence[2)]; the rapid dependence of the tree- or 1-loop EP, $V^{(0)}$ or $V^{(1)}$, on the choice of renormalization points μ and T_0 is the most popular example. No reliable prediction can be made without solving this problem[3)].

We know that the exact EP at finite temperature satisfies a set of two renormalization group equations[4)] (RGEs), whose differential operators are just total derivatives with respect to μ and T_0. Namely, the exact EP is automatically μ- and T_0-independent. How can we use this fact to solve the problem of RS-dependence, especially of the renormalization scale dependence? This is the key question for carrying out the RG improvement of the EP. Recently, in the case of the zero-temperature field theory, an elegant procedure has been proposed[5)] to improve the EP so as to satisfy the RGE.

In this paper we extend the idea of Ref. 5) to the finite temperature field theory and propose a simple and effective procedure for improving the finite temperature EP so as to satisfy the corresponding RGEs.

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II. RG improvement at finite temperature

For definiteness, let us consider the $O(N)$ symmetric massive $\lambda\phi^4$ model of an N-component real scalar field in the large-N limit. The Lagrangian density of the system is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{8}\lambda(\phi^2)^2 - hm^4 , \quad (1)$$
$$\phi^2 = \phi^a\phi^a, \quad a = 1,\ 2,\ \cdots,\ N .$$

Suppose that we employ the mass-independent renormalization and renormalize the theory at an arbitrary mass-scale μ and at an arbitrary temperature T_0 with definite renormalization prescriptions (e.g., the modified MOM scheme with symmetric/asymmetric renormalization, etc.). Here we pay attension only to the μ- and T_0-dependences of the perturbative result.

Then the effective potential (EP) satisfies the two renormalization group equations (RGEs) with respect to the renormalization points μ and $\xi(\equiv T_0/\mu)$:[6]

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\mu\frac{\partial}{\partial\lambda} - m^2\theta_\mu\frac{\partial}{\partial m^2} - \phi\gamma_\mu\frac{\partial}{\partial\phi} + \beta_{h\mu}\frac{\partial}{\partial h}\right) V(\phi, m^2, \lambda, h, T; \mu^2, \xi^2) = 0 , \quad (2)$$

$$\left(\xi\frac{\partial}{\partial\xi} + \beta_\xi\frac{\partial}{\partial\lambda} - m^2\theta_\xi\frac{\partial}{\partial m^2} - \phi\gamma_\xi\frac{\partial}{\partial\phi} + \beta_{h\xi}\frac{\partial}{\partial h}\right) V(\phi, m^2, \lambda, h, T; \mu^2, \xi^2) = 0 . \quad (3)$$

The solution is given by

$$V(\phi, m^2, \lambda, h, T; \mu^2, \xi^2) = V(\bar\phi(t,\rho), \bar m^2(t,\rho), \bar\lambda(t,\rho), \bar h(t,\rho), T; \mu^2 e^{2t}, \xi^2 e^{2\rho}), \quad (4)$$

where $\bar\phi$, $\bar m^2$, $\bar\lambda$ and $\bar h$ are running parameters whose t- and ρ-dependences are determined by

$$\frac{\partial\bar\lambda}{\partial x} = \beta_i(\bar\lambda),\quad \frac{\partial\bar m^2}{\partial x} = -\theta_i(\bar\lambda)\bar m^2,$$
$$\frac{\partial\bar\phi}{\partial x} = -\gamma_i(\bar\lambda)\bar\phi,\quad \frac{\partial\bar h}{\partial x} = \beta_{hi}(\bar h, \bar\lambda),\quad (x,i) = (t,\mu),(\rho,\xi), \quad (5)$$

with the boundary condition that the corresponding barred quantities reduce to the unbarred parameters at $t = \rho = 0$. Therefore, *the EP is completely determined once its function form is known at certain values of t and ρ.*

Then the question posed in the introduction reduces to the following: How can we determine, with the limited knowledge of the L-loop calculations, the functional form of the EP?

In case of the zero-temperature field theory, by studying the logarithmic structure of the L-loop EP, $V^{(L)}$, Bando et al.[5] showed that the knowledge up to the L-loop calculations can determine the functional form of the EP being exact up to the Lth-to-leading log order. Beautiful point of their procedure is that the EP thus determined automatically satisfies the RGE (2) up to the Lth-to-leading log accuracy level.

In order to extend the idea of Ref. 5) to the $T \neq 0$ case of our interest, here we study the structure of the perturbatively evaluated EP. In the large-N limit, $Z_\phi = 1$ (namely, $\gamma_\mu = \gamma_\xi = 0$) and contributed diagrams are chain types, daisy types and super-daisy types only, which have the following general structures:

1) contribution from chain diagrams ($L \geq 2$):

$$V^{(L)}_{chain} = \frac{NM^4}{\lambda_N} \text{[polynomials in } \Delta_1 \text{ , } \Delta_2] \quad , \tag{6}$$

2) contribution from daisy diagrams ($L \geq 4$):

$$V^{(L)}_{daisy} = \frac{NM^4}{\lambda_N} \left[(\text{numerical factor}) \times (\lambda_N \Delta_1)^{L-1} \Delta_{L-1} \right] \quad , \tag{7}$$

3) contribution from super-daisy diagrams ($L \geq 5$):

$$V^{(L)}_{super-daisy} = \frac{NM^4}{\lambda_N} \text{[polynomials in } \Delta_1 \text{ , } \cdots \text{ ,} \Delta_{L-2}] \quad , \tag{8}$$

where $\lambda_N \equiv N\lambda$, $M^2 \equiv m^2 + (1/2)\lambda\phi^2$ and

$$M^2 \Delta_1 \equiv \frac{i}{2} \int_k \frac{1}{k^2 - M^2} + (\delta M^2)^{(1)}, \tag{9}$$

$$\Delta_2 \equiv \frac{i}{2} \int_k \frac{1}{(k^2 - M^2)^2} + Z^{(1)}_\lambda, \tag{10}$$

$$\frac{1}{(M^2)^{n-2}} \Delta_n \equiv \frac{i}{2} \int_k \frac{1}{(k^2 - M^2)^n} \quad (n \geq 3) \; . \tag{11}$$

$(\delta M^2)^{(1)}$ and $Z^{(1)}_\lambda$ are the 1-loop mass and coupling counterterms respectively, and $\int_k$ denotes the k_0-summation and $\boldsymbol{k}$-integration. Thus by introducing an effective variable $\tau \equiv \lambda_N \Delta_1$, the L-loop contribution to the EP can be expressed in the power-series in τ;

$$V^{(L)} = \frac{NM^4}{\lambda_N} \left[\sum_{\ell=0}^{L} \lambda_N^\ell v_\ell^{(L)} \tau^{L-\ell} + z\delta_{L,0} \right] \; , \quad z \equiv \lambda_N h \frac{m^4}{M^4} \; . \tag{12}$$

Then, the full EP becomes

$$V = \sum_{L=0}^{\infty} V^{(L)} = \frac{NM^4}{\lambda_N} \sum_{\ell=0}^{\infty} \lambda_N^\ell \left[F_\ell(\tau) + z\delta_{\ell,0} \right] \; , \tag{13}$$

where

$$F_\ell(\tau) \equiv \sum_{L=\ell}^{\infty} v_\ell^{(L)} \tau^{L-\ell}. \tag{14}$$

This form of expansion (13) in powers of λ_N just gives a '*leading-τ*' series expansion: namely, F_0, F_1, ... correspond to the '*leading*', '*next-to-leading*', ... τ terms, respectively. The meaning of '*leading-τ*' becomes clear later.

At $\tau = 0$, the '*ℓth-to-leading* τ' function F_ℓ is given solely in terms of the ℓ-loop level potential, $F_\ell(\tau = 0) = v_\ell^{(L=\ell)}$. So, if we calculated the EP up to the L-loop level $V_L = V^{(0)} + V^{(1)} + \cdots + V^{(L)}$, then at $\tau = 0$ it already gives the function '*exact*' up to '*Lth-to-leading* τ' order:

$$V = \frac{NM^4}{\lambda_N} \sum_{\ell=0}^{L} \lambda_N^\ell \left[v_\ell^{(L=\ell)} + z\delta_{\ell,0} \right] \;+\; O(\lambda_N^L) = V_L|_{\tau=0} \;+\; O(\lambda_N^L) \,. \tag{15}$$

Therefore, with the L-loop potential V_L at hand, the EP satisfying the RGEs can be given by

$$\begin{aligned} V &= N\bar{M}^4(t) \sum_{\ell=0}^{L} \bar{\lambda}_N^{\ell-1}(t,\rho) \left[\bar{v}_\ell^{(\ell)}(t,\rho) + \bar{z}(t,\rho)\delta_{\ell,0} \right] \Bigg|_{\bar{\tau}(t,\rho)=0} \\ &= V_L\left(\phi, \bar{m}^2(t,\rho), \bar{\lambda}_N(t,\rho), \bar{h}(t,\rho); \mu^2 e^{2t}, \xi^2 e^{2\rho}\right)\Big|_{\bar{\tau}(t,\rho)=0} \,, \end{aligned} \tag{16}$$

where the barred quantities should be evaluated at t and ρ satisfying

$$\bar{\tau}(t,\rho) \equiv \bar{\lambda}_N(t,\rho)\bar{\Delta}_1(t,\rho) = 0.$$

Although the solution (16) is "exact" only up to Lth-to-leading τ order, it satisfies the RGEs exactly if the runnings of the barred quantites are solved exactly. If the runnings of the parameter $\bar{\lambda}_N/\lambda_N$, $\bar{m}^2/m^2$ and $\bar{h}/h$ are solved correctly only up to Lth power in λ_N in the sense of the leading τ expansion, our solution (16) satisfies the RGE up to Lth-to-leading τ order and "exact" in that order.

III. Explicit calculations

In this section, we demonstrate our procedure by explicit calculations to the '*next-to-leading* τ' order ($L = 2$). Here, for simplicity, we use the zero temperature renormalization scheme, and thus all renormalization constants are the same as those at $T = 0$.

The unimproved effective potential is

$$\begin{aligned} V_2 &= \frac{NM^4}{2\lambda_N} \left[1 + \tau + \tau^2 + \lambda_N \left\{ -\frac{1}{64\pi^2} + \frac{T^4}{\pi^2 M^4} L_0\left(\frac{T^2}{M^2}\right) - \frac{T^2}{2\pi^2 M^2} L_1\left(\frac{T^2}{M^2}\right) \right\} \right. \\ &\qquad \left. + \lambda_N^2 \left(\frac{1}{32\pi^2}\right)^2 \right] + m^4\left(h - \frac{N}{2\lambda_N}\right) \,, \end{aligned} \tag{17}$$

where

$$\tau \equiv \lambda_N \left\{ \frac{1}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} - 1\right) + \frac{T^2}{2\pi^2 M^2} L_1\left(\frac{T^2}{M^2}\right) \right\} \,, \tag{18}$$

$$L_0\left(\frac{1}{a^2}\right) \equiv \int_0^\infty k^2\, dk\, \ln[1 - \exp\{-\sqrt{k^2 + a^2}\}] \,, \tag{19}$$

$$L_1\left(\frac{1}{a^2}\right) \equiv \frac{1}{2}\int_0^\infty \frac{k^2\, dk}{\sqrt{k^2 + a^2}} \frac{1}{\exp\{\sqrt{k^2 + a^2}\} - 1} \,. \tag{20}$$

The coefficient functions of the RGE are the same as those at $T = 0$ and calculated by (exact up to 2-loop level)

$$\beta_\mu(\lambda_N) = \frac{1}{16\pi^2}\lambda_N^2 \equiv b\lambda_N^2 \ , \quad \theta_\mu(\lambda_N) = -b\lambda_N \ , \quad \beta_{h\mu}(h, \lambda_N) = \frac{Nb}{2} - 2bh\lambda_N \ . \tag{21}$$

Thus, we can get the improved EP as follows:

$$\begin{aligned} \bar{V}_2 \ = \ & \frac{N\bar{M}^4(t)}{2\bar{\lambda}_N(t)}\left[1 + \bar{\lambda}_N(t)\left\{-\frac{1}{64\pi^2} + \frac{T^4}{\pi^2\bar{M}^4(t)}L_0\left(\frac{T^2}{\bar{M}^2(t)}\right)\right.\right. \\ & \left.\left. -\frac{T^2}{2\pi^2\bar{M}^2(t)}L_1\left(\frac{T^2}{\bar{M}^2(t)}\right)\right\} + \bar{\lambda}_N^2(t)\left(\frac{1}{32\pi^2}\right)^2\right] + \bar{m}^4(t)\left(\bar{h}(t) - \frac{N}{2\bar{\lambda}_N(t)}\right) \end{aligned} \tag{22}$$

where

$$\bar{M}^2(t) = \frac{M^2}{1 - tb\lambda_N}, \quad \bar{\lambda}_N(t) = \frac{\lambda_N}{1 - tb\lambda_N} \ , \tag{23}$$

and the barred quantities are evaluated at such a t that satisfies $\bar{\tau}(t) = 0$.

Here, we make clear the meanings of '*leading-τ*' and '*exact*'. τ includes leading log term and leading T term for large T:

$$\tau \simeq \lambda_N\left[\frac{1}{32\pi^2}\ln\frac{M^2}{\mu^2} + \frac{T^2}{24\pi M^2}\right] . \tag{24}$$

Therefore, we use the term '*ℓth-to-leading τ*' in the simultaneous sense of the ℓth-to-leading log and the ℓth-to-leading T. Namely, " '*exact*' up to '*Lth-to-leading τ*' order " means " *exact* up to *Lth-to-leading log* as well as *Lth-to-leading* T order".

Next, let us see the behavior of the improved EP.

- Expanding the improved $\bar{V}_1$ with respect to (unbarred coupling) λ_N, we get

$$\begin{aligned} \bar{V}_1 \ = \ & \frac{NM^4}{2\lambda_N}\left[1 + \lambda_N\left\{\frac{1}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\right) + \frac{T^4}{\pi^2M^4}L_0\left(\frac{T^2}{M^2}\right)\right\}\right. \\ & \left. + \lambda_N^2\left(\left\{\frac{1}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} - 1\right) + \frac{T^2}{2\pi^2M^2}L_1\left(\frac{T^2}{M^2}\right)\right\}^2 + \left(\frac{1}{32\pi^2}\right)^2\right)\right] \\ & + m^4\left(h - \frac{N}{2\lambda_N}\right) \ . \end{aligned} \tag{25}$$

  This is nothing but the unimproved V_2, Eq.(17).
- The condition $\bar{\tau}(t) = 0$, which determines the function form of the EP up to the disired next-to-leadinf τ accuracy level, gives the equation

$$\bar{M}^2(t) = M_L^2 - \frac{\lambda_N T}{8\pi}\bar{M}(t) + \frac{1}{2}\lambda_N bA\bar{M}^2(t) + \cdots \quad , \tag{26}$$

  where

$$M_L^2 \ = \ M^2 + \frac{\lambda_N T}{24} \quad , \tag{27}$$

$$A \ = \ 2\left(\log\frac{2\pi T}{\mu} - \gamma_E\right) \ . \tag{28}$$

Eq.(26) is valid in the high temperature regime, and works as the mass gap equation which determines the $\bar{M}^2(t)$ at high temperature.
$\bar{M}(t)$ can be expanded for small λ_N as

$$\bar{M}(t) = M_L \left\{ 1 - \frac{\lambda_N T}{16\pi M_L} + O\left(\left(\frac{\lambda_N T}{16\pi M_L}\right)^2\right)\right\} \quad . \tag{29}$$

- We calculate also the critical temperature T_C in a crude analytic estimate:

$$m^2 + \frac{\lambda_N T_C^2}{24} = 0 \ . \tag{30}$$

This agrees with the result of Dolan-Jackiw[7].

IV. Summary

- We proposed a *simple* and *effective* procedure of the RG improvement of the effective potential at finite temperature.
- Applied it to the perturbative calculation in zero-temperature renormalization scheme: it automatically carried out the large-log resummation as well as the resummation of large-T terms through the chain, daisy and super-daisy summations.
- This procedure may be extensible to general case by studying the structure of perturbative expansion.

Footnotes and References

* This paper is based on the talk given at the 3rd Workshop on Thermal Field Theories and Their Applications (15–27 August, 1993, Banff, Canada). Results of more extensive analysis will be published elsewhere.

1) R. Jackiw and G. Amelino-Camelia, in *BANFF/CAP Workshop on THERMAL FIELD THEORY*, Proceedings of the 3rd Workshop on Thermal Field Theories and Their Applications (15-27 August, 1993, Banff, Canada), edited by F. C. Khanna et al. (World Scientific), p.180.

2) See, e.g., T. Muta, *Foundations of Quantum Chromodynamics* (World Scientific, Singapore, 1987).

3) The renormalization-scheme (RS) is nothing but a precise precription to define the renormalization constant. In this sense, the renormalization scale (point) μ is *one of* the parameters that specify the RS. For details of the RS-dependence, see, e.g. Ref. 2).

4) H. Matsumoto, Y. Nakano and H. Umezawa, *Phys. Rev.* D29 (1984) 1116.

5) M. Bando, T. Kugo, N. Maekawa and H. Nakano, *Phys. Lett. B* 301 (1993) 83.

6) The choice of μ and ξ as independent parameters is preferable to the other choice of μ and T_0, since with this choice the beta- and gamma-functions appearing in RGE (2) have clear correspondence with the zero-temperature counterparts.

7) L. Dolan and R. Jackiw, *Phys. Rev.* D9 (1974) 3320.