Resolution of Scheme-Dependences of the Photon Structure Function

—Solution to the Problem of Large Corrections in the Large-n Limit—

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Optimization analysis of the photon structure function is worked out for a simplified two-component model tbat keeps the essential behavior of the complete structure function and becomes exact in the large-n limit. Our findings are as follows. i) Thorough resolution of scheme-dependences can be achieved satisfactorily if the renormalization and the factorization scheme dependences are investigated simultaneously. ii) The two-scale formalism of optimized perturbation theory has dealt with the large second order corrections as $n\to\infty$ so that at least part of the leading large-n terms are absorbed into the coupling constant, and that the optimized perturbation expansion may show an improved convergence behavior, thus giving a solution to the problem of large corrections in the large-n limit. iii) The above solution agrees at least up to the leading large-n terms with the results obtained through the kinematical analyses based on the resummation of singular terms. iv) The optimal scheme, in which we can get the above beautiful consequences, has been shown exactly to lie within a class of schemes where two-loop hadronic anomalous dimension vanishes.

I. Introduction

The asymptotically dominant contribution to the photon structure function comes from the inhomogeneous point-like term that can be calculated solely from quantum chromodynamics (QCD) without suffering from any unknown matrix elements of local operators[1),2)], and the subdominant hadron-like (or VMD-like) term contributes essentially only to the small-x regions or the small-n moments[3)]. Because of this beautiful fact the photon structure function has been discussed as a clean testing ground of QCD. However, once we apply QCD perturbation theory the photon structure function can not either escape the famous two diseases: The first one is the calculational scheme-dependences[4)], such as the renormalization scheme (RS) dependence[5),6),7)] and the factorization scheme (FS) dependence[6),8),10)], of the result inherent in QCD perturbation theory, and the second one is the large negative subleading order corrections coming from singular terms for $x\sim 1$[11),12)], or equivalently

as $n\to\infty$ that might be of kinematical origin.

In order to study the problem of scheme-dependences the key observation is the following fact: Perturbative QCD calculations of physical quantities depend essentially on two calculational schemes[4),18)], the RS and FS, and because the FS-dependence is closely connected with the RS-dependence, both scheme-dependences should be analyzed simultaneously[9),10)]. Resolution of ambiguities due to scheme-dependences has been carried out successfully[10)] with the use of the generalized two-scale formalism[10)] of optimized perturbation theory (OPT) based on the principle of minimal sensitivity (PMS)[7)]. Analysis of hadronic structure function has provided us with, among others, an interesting consequence about the convergence behavior of perturbation expansion[10)]; Through the optimization part of the large corrections are absorbed into the optimized effective mass-scale with which the coupling constant runs, and the optimized perturbation coefficients become moderate. Thus the convergence of the perturbation expansion in terms of the coupling constant is formally improved.

This fact reminds us of the consequence of kinematical approaches[14),15)] to handle the large subleading order corrections coming from kinematically singular terms mentioned above. Amati *et al*[15)] claimed with *a posteriori* justification[16)] that, by resumming such perturbation series being singular as $x\to 1$ (or $n\to\infty$) to all orders of the strong coupling constant, the argument of the effective coupling constant at each vertex of the ladder rung is rescaled as $Q^2\to Q^2(1-x)$ and that the perturbation expansion in terms of such a "rescaled" coupling constant contains only less singular perturbation coefficients thus shows formally an improved behavior of convergence. It is interesting to carry out the optimization for various quantities and to compare the results with those from the kinematical analyses. Such investigation may shed some light on the question what the PMS-optimization does, not in the context of a mathematical prescription but as a physical effect.

Recently based on the analyses of structure functions of the photon[17)] as well as hadrons[10)] we have shown[18)] that the optimization through the two-scale formalism may in fact have correctly taken into account the kinematical boundary effects, and that a kind of "equivalence" holds for the treatment of kinematically singular higher order perturbation coefficients between the OPT based on PMS and the kinematical approaches. In contrast to hadronic structure functions, however, the OPT analysis of the photon structure function is quite complicated, as has been discussed in Ref. 17, and the complete analysis can be carried out only through numerical computations with one important assumption. Therefore the analysis of the photon case should be supplemented with further investigations based on a firm foundation. For this purpose we consider in this paper a simplified model for the photon structure function and carry out the optimization exactly without any assumptions. Because we are interested in the problem how the kinematically singular higher order coefficients are dealt with in connection with the kinematical approaches, the model considered should keep the essense of the complete photon

structure function in the large-n limit. This analysis also gives a justification for the assumption adopted in Ref. 17.

II. Structure Function of the Photon

A. The model

The n-th moment of the photon structure function

$$F_n^\gamma(Q^2)=\int_0^1 dx\ x^{n-2}\ F_2^\gamma(x,\ Q^2) \tag{1}$$

can be factorized in terms of the Wilson's operator product expansion (OPE) as

$$F_n^\gamma(Q^2)=\vec{\vartheta}_n(a(M))\cdot\ \vec{\mathscr{C}}_n\ (Q/\mu,\ M/\mu,\ \bar{a}(\mu)), \tag{2}$$

where ϑ_n is the matrix element of the local operator with spin-n, and $\mathscr{C}_n$ the OPE coefficient function. Throughout this paper we carry out the calculation valid to the first order in the electromagnetic fine structure constant $\alpha_{em}=e^2/4\pi$. Following the two-scale formalism of OPT[10] we calculate the OPE coefficient function in terms of the QCD effective coupling constant $\bar{a}(\mu)=\bar{g}^2(\mu)/4\pi^2$, renormalized at the scale μ, and renormalize the operator matrix element at the scale M.

In this paper we consider a simplified two-component model of the photon structure function, namely in Eq. (2) $\vec{\vartheta}$ and $\vec{\mathscr{C}}$ are two-component row and column vectors, respectively (hereafter the moment-index n is suppressed for simplicity),

$$\vec{\vartheta}\ =\ (\vartheta_{NS},\ \vartheta_\gamma), \tag{3a}$$

$$\vec{\mathscr{C}}\ =\ \begin{pmatrix}\mathscr{C}_{NS}\\ \mathscr{C}_\gamma\end{pmatrix}\ , \tag{3b}$$

or

$$F^\gamma\ =\ \vec{\vartheta}\cdot\ \vec{\mathscr{C}}=\vartheta_{NS}\ \mathscr{C}_{NS}\ +\ \vartheta_\gamma\ \mathscr{C}_\gamma. \tag{3c}$$

The operator matrix element $\vec{\vartheta}$ satisfies the equation

$$\frac{d\ \vec{\vartheta}(a(M))}{d\ \ln\ M}\ =\ \vec{\vartheta}(a(M))\cdot\gamma(a(M)), \tag{4}$$

where γ denotes the 2 by 2 anomalous dimension matrix

$$\gamma\ =\ \begin{pmatrix}\gamma_{NS} & 0\\ K_{NS} & 0\end{pmatrix} \tag{5}$$

Then Eq.(4) is decomposed into two equations

$$\frac{d\ \vartheta_{NS}(a(M))}{d\ \ln\ M}\ =\ \vartheta_{NS}(a(M))\gamma_{NS}(a(M))+\vartheta_\gamma(a(M))K_{NS}(a(M)), \tag{6a}$$

$$\frac{d\ \vartheta_\gamma(a(M))}{d\ \ln\ M}\ =\ 0,\ \text{or}\ \vartheta_\gamma\ \text{is independent of M}, \tag{6b}$$

which can be easily integrated to give[10]

$$\vartheta_{NS}(a(M))\ =\ A\ \exp\int_{a_0}^{a(M)} dx\ \frac{\gamma_{NS}(x)}{\beta(x)}$$

$$+\ \vartheta_\gamma\int_1^{a(M)} dx\ \frac{K_{NS}(x)}{\beta(x)}\ \exp\int_x^{a(M)} dy\ \frac{\gamma_{NS}(y)}{\beta(y)}\ , \tag{7}$$

where A is the scheme-independent constant[9),10)], and the β-functions are defined as[4),13)]

$$d\tilde{a}(\mu)/d\ln\mu=\tilde{\beta}(\tilde{a}(\mu)),\ \tilde{a}(\mu=\mu_0)\ =\ g_\mu^2\ /\ 4\pi^2, \tag{8a}$$

$$da(M)/d\ln M=\beta(a(M)),\ a(M=M_0)=g_M^2/4\pi^2. \tag{8b}$$

As usually[2] we shall neglect the first term in Eq.(7), keeping only the inhomogeneous point-like term that is dominant in the asymptotic limit. Thus we are not concerned with the structure function at small-x regions[3].

Perturbation expansions for various quantities are as follows;

$$\begin{aligned}
\gamma_{NS}(x) &= \gamma_{NS}^0\ x+\gamma_{NS}^1\ x^2+\cdots &,\\
K_{NS}(x) &= e^2(K_{NS}^0+K_{NS}^1\ x+\cdots) &,\\
\beta(x) &= bx^2(1+cx+c_2\ x^2+\cdots) &,\\
\tilde{\beta}(x) &= bx^2(1+cx+\tilde{c}_2\ x^2+\cdots) &,\\
\mathscr{C}_{NS}(x) &= e^2\ \delta_{NS}(1+B_{NS}\ x+\cdots) &,\\
\mathscr{C}_\gamma(x) &= e^4\ \delta_\gamma(B_\gamma+\cdots) &,\\
\vartheta_\gamma &= 1 &,
\end{aligned} \tag{9}$$

where x denotes the effective coupling constant $a(M)$ or $\tilde{a}(\mu)$. It is to be noted that the leading and the second order coefficients of the β-functions, b and c, and the leading order anomalous dimension coefficients γ_{NS}^0 and K_{NS}^0 are all scheme-independent quantities, which have been calculated already[2,3]. Other second order coefficients γ_{NS}^1, K_{NS}^1, B_{NS} and B_γ depend on schemes used in the actual calculations.

Now consider the next-to-leading (second) order approximation by truncating everywhere, e. g., in Eqs.(9), the perturbation expansion up to the second order terms. In this order two β-functions β and $\tilde{\beta}$ coincide. Then we get the second order expression for the structure function as

$$\begin{aligned}
\hat{F}^\gamma &\equiv -bF^\gamma/e^4\\
&= -b\delta_\gamma B_\gamma+\delta_{NS}(1+B_{NS}\ \tilde{a})\left(\frac{ca}{1+ca}\right)^{d_{NS}}(1+ca)^{d_{NS}^1}\\
&\quad\times\int_{\frac{ca}{1+ca}}^{\frac{c}{1+c}} dz\left\{cK_{NS}^0\ z^{-d_{NS}-2}(1-z)^{d_{NS}^1+1}\ +\ K_{NS}^1\ z^{-d_{NS}-1}(1-z)^{d_{NS}^1}\right\},
\end{aligned} \tag{10}$$

where $a\equiv a(M)$, $\tilde{a}\equiv\tilde{a}(\mu)$, $d_{NS}\equiv\gamma_{NS}^0/b$ and $d_{NS}^1\equiv\gamma_{NS}^1/bc$. Eq.(10) is the "exact" expression up to the second order calculations. In the limit of small coupling constant $a\simeq\tilde{a}\simeq 0$, we can get a more convenient expression. In taking into account the fact that the upper limit of the integration in (10), $c/(1+c)$, is less than unity, or

$$c/(1+c)\simeq 0.606 \quad \text{for } n_f=4,$$

where n_f is the number of quark flavors, we understand that the main contribution to the integral comes from the lower limit. Thus by expanding the integrand around $z=0$ and keeping up to the desired terms, we get the following "approximate" expression

$$\hat{F}_\tau = -b\delta_\tau B_\tau + \delta_{NS}(1+B_{NS}\tilde{a})(1+ca)^{d^1_{NS}}$$
$$\times\left[\frac{K^0_{NS}}{1+d_{NS}}\,\frac{1+ca}{a}+\frac{K^1_{NS}-cK^0_{NS}(1+d^1_{NS})}{d_{NS}}\right.$$
$$\left.+\frac{cd^1_{NS}}{d_{NS}-1}\left\{\frac{1}{2}cK^0_{NS}(1+d^1_{NS})-K^1_{NS}\right\}\frac{a}{1+ca}\right]. \tag{11}$$

To get the above expression we have neglected terms proportional to $(ca/(1+ca))^{d_{NS}}$, to be consistent with the approximation that we have neglected the first term in Eq. (7).

The second order effective coupling constants $a(M)$ and $\tilde{a}(\mu)$ are defined by integrating the renormalization group equations (8) as[7),10)]

$$|b|\ln\frac{\mu}{\tilde{\Lambda}}=\int_{\tilde{a}}^{\infty}\frac{dx}{x^2(1+cx)}=\frac{1}{\tilde{a}}+c\ln\left(\frac{c\tilde{a}}{1+c\tilde{a}}\right), \tag{12a}$$

$$|b|\ln\frac{M}{\tilde{\Lambda}}=\int_{a}^{\infty}\frac{dx}{x^2(1+cx)}=\frac{1}{a}+c\ln\left(\frac{ca}{1+ca}\right). \tag{12b}$$

B. Connection with the complete structure function

Here we discuss about the connection of the simplified model presented in A with the complete photon structure function, and show that in the large-n limit the complete structure function essentially reduces to the model considered.

The complete photon structure function consists of four components, i. e.,

$$\vec{\vartheta} = (\vartheta_\phi,\ \vartheta_G,\ \vartheta_{NS},\ \vartheta_\tau) = (\hat{\vartheta},\ \vartheta_\tau), \tag{13a}$$

$${}^t(\vec{\mathscr{C}}) = (\mathscr{C}_\phi,\ \mathscr{C}_G,\ \mathscr{C}_{NS},\ \mathscr{C}_\tau) = ({}^t\hat{\mathscr{C}},\ \mathscr{C}_\tau), \tag{13b}$$

$\hat{\vartheta}$ and $\hat{\mathscr{C}}$ are three-component vectors whose components are obvious. The operator matrix element $\vec{\vartheta}$ satisfies the equation

$$\frac{d\,\vec{\vartheta}(a(M))}{d\ln M} = \vec{\vartheta}(a(M))\cdot\gamma(a(M)), \tag{14}$$

where the anomalous dimension matrix γ is given by

$$\gamma = \left(\begin{array}{ccc:c} \gamma_{\phi\phi} & \gamma_{G\phi} & 0 & 0 \\ \gamma_{\phi G} & \gamma_{GG} & 0 & 0 \\ 0 & 0 & \gamma_{NS} & 0 \\ \hdashline K_\phi & K_G & K_{NS} & 0 \end{array}\right) \equiv \left(\begin{array}{c:c} \hat{\gamma} & 0 \\ \hdashline \hat{K} & 0 \end{array}\right). \tag{15}$$

Eq.(14) can be integrated to give[19)]

$$\hat{\vartheta}(a(M)) = A\ \mathrm{T}\exp\int_{a_0}^{a(M)}dx\,\frac{\hat{\gamma}(x)}{\beta(x)}$$
$$+\vartheta_\tau\int_1^{a(M)}dx\,\frac{\hat{K}(x)}{\beta(x)}\ \mathrm{T}\exp\int_x^{a(M)}dy\,\frac{\hat{\gamma}(y)}{\beta(y)}, \tag{16a}$$

$$\vartheta_\tau = 1. \tag{16b}$$

The T-ordering in Eq. (16a) is now necessary because of the matrix structure of the anomalous dimension matrix $\hat{\gamma}$, namely, $[\hat{\gamma}(a'),\ \hat{\gamma}(a'')] \neq 0$.

The existing second order calculations in the MS and/or $\overline{\mathrm{MS}}$ scheme show, however, that the one- and two-loop anomalous dimensions behave in the large-n

limit as follows[20),21)];

$$\gamma^0_{\phi\phi} \sim \gamma^0_{GG} \sim \gamma^0_{NS} \sim \mathcal{O}(\ln n), \tag{17a}$$

$$\gamma^0_{\phi G} \sim \gamma^0_{G\phi} \sim \mathcal{O}(1/n), \tag{17b}$$

$$\gamma^1_{\phi\phi} \sim \gamma^1_{GG} \sim \gamma^1_{NS} \sim \mathcal{O}(\ln n), \tag{17c}$$

$$\gamma^1_{\phi G} \sim \gamma^1_{G\phi} \sim \mathcal{O}(\ln^2 n/n). \tag{17d}$$

Thus in the limit $n \to \infty$ the anomalous dimension matrix $\hat{\gamma}$ in Eqs.(15) and (16) becomes essentially a diagonal matrix and the T-ordering in Eq.(16) can be omitted. Therefore we get (the first term in (16a) is neglected)

$$\begin{aligned} F_T/e^4 = {} & \delta_\phi \vartheta_\phi(a(M))(1+B_\phi \tilde{a}(\mu)) \\ & + \delta_\phi \vartheta_G(a(M)) B_G \tilde{a}(\mu) \\ & + \delta_{NS} \vartheta_{NS}(a(M))(1+B_{NS}\tilde{a}(\mu)) \\ & + \delta_T B_T , \end{aligned} \tag{18a}$$

$$\vartheta_i(a(M)) = \int_1^{a(M)} dx \, \frac{K_i^0 + K_i^1 x}{bx^2(1+cx)} \exp \int_x^{a(M)} dy \, \frac{\gamma_i^0 + \gamma_i^1 y}{by(1+cy)} , \quad (i=\phi,\ G,\ NS) . \tag{18b}$$

If we further take into account the facts that the gluon contribution (the second term in (18a)) is not effective in the limit $n\to\infty$ (or $x\to 1$), and that the fermionic contribution is essentially the same as the non-singlet contribution, then we recognize that the model presented in A is essentially nothing but the large-n limit of the complete photon structure function.

III. Optimization

A. Optimization variables

Now we consider the second order expression of the model structure function, Eq. (10) or Eq. (11), and apply the two-scale formalism of OPT. In the second order approximation, a set of parameters that label scheme-dependences is [the renormalization scale $\ln\mu$, the factorization scale $\ln M$, the two-loop anomalous dimension $d^1_{NS} \equiv \gamma^1_{NS}/bc$ and K^1_{NS}], and the quantities to be optimized are the second order coefficients B_{NS}, B_T and the effective coupling constants $a(M)$ and $\tilde{a}(\mu)$. In the following we take all the above four scheme-labeling parameters ($\ln\mu$, $\ln M$, d^1_{NS} and K^1_{NS}) as independent optimization variables.

Before entering the details of the optimization of the photon structure function, we here take notice of the following fact: The perturbation coefficient B_{NS} appearing in Eqs. (10) and (11) is nothing but that appears in the calculation of the non-singlet component of hadron structure function. Thus the structure of B_{NS}, except its optimal value, should be determined through the analysis of hadron structure function. Such analysis has already deen done[10] and the resulting equations coming from the consistency requirement of perturbation theory are

$$\partial B_{NS} / \partial \ln\mu = 0, \tag{19a}$$

$$\partial B_{NS} / \partial \ln M = -b\, d_{NS} \ (\equiv -\gamma^0_{NS}), \tag{19b}$$

$$\partial B_{NS} / \partial d_{NS}^1 = -c. \tag{19c}$$

Obviously B_{NS} should be independent of the anomalous dimension K_{NS}^1 that appears only in relation to the photon structure function, i. e.,

$$\partial B_{NS} / \partial K_{NS}^1 = 0. \tag{20}$$

In order to carry out the optimization procedure we should at first evaluate the response of the structure function (10) or (11) to changes of the scheme-labeling parameters. We shall discuss the exact optimization starting from the "exact" structure function up to the second order calculation, Eq.(10), and the optimization in the small coupling limit starting from Eq.(11), separately.

B. Exact optimization

The response of the structure function (10) to changes of $\ln\mu$ and K_{NS}^1 is given as follows;

$$\frac{\partial \hat{F}_T}{\partial \ln\mu} = -b\delta_T \frac{\partial B_T}{\partial \ln\mu} + \delta_{NS} B_{NS} b\tilde{a}^2(1+c\tilde{a})\left(\frac{ca}{1+ca}\right)^{d_{NS}}(1+ca)^{d_{NS}^1}$$

$$\times \int_{\frac{ca}{1+ca}}^{\frac{c}{1+c}} dz \left\{ cK_{NS}^0 z^{-d_{NS}-2}(1-z)^{d_{NS}^1+1} + K_{NS}^1 z^{-d_{NS}-1}(1-z)^{d_{NS}^1} \right\}, \tag{21a}$$

$$\frac{\partial \hat{F}_T}{\partial K_{NS}^1} = -b\delta_T \frac{\partial B_T}{\partial K_{NS}^1} + \delta_{NS}(1+B_{NS}\tilde{a})\left(\frac{ca}{1+ca}\right)^{d_{NS}}(1+ca)^{d_{NS}^1}$$

$$\times \int_{\frac{ca}{1+ca}}^{\frac{c}{1+c}} dz\, z^{-d_{NS}-1}(1-z)^{d_{NS}^1}, \tag{21b}$$

where we have made use of Eqs.(19a) and (20). It is to be noted that the integrals in Eqs. (21a) and (21b) are of the order $\mathcal{O}(a^{-d_{NS}-1})$ and $\mathcal{O}(a^{-d_{NS}})$, respectively. Responses with respect to other variables $\ln M$ and d_{NS}^1 are complicated and it is difficult to get useful conclusions. Therefore we here study consequences obtained from Eqs.(21).

According to the optimization procedure, we then impose that the response of $\hat{F}_T$, Eqs.(21), to be of order a (the consistency requirement of perturbation theory), and obtain consistency equations

$$\frac{\partial B_T}{\partial \ln\mu} = 0, \tag{22a}$$

$$\frac{\partial B_T}{\partial K_{NS}^1} = -\frac{\delta_{NS}}{b\delta_T d_{NS}}. \tag{22b}$$

Finally we get the optimization equations by setting the variations (21a, b) to be exactly zero after the substitution of consistency equations (22a, b),

$$B_{NS} = 0, \text{ (exact)}, \tag{23a}$$

$$d_{NS}\left(\frac{ca}{1+ca}\right)^{d_{NS}}(1+ca)^{d_{NS}^1}\int_{\frac{ca}{1+ca}}^{\frac{c}{1+c}} dz\, z^{-d_{NS}-1}(1-z)^{d_{NS}^1} = 1. \tag{23b}$$

The solution to Eq.(23b) is

$$d_{NS}^1 = 0, \tag{24}$$

which is exact up to the hadronic contributions of the order $\mathcal{O}(a^{d_{NS}})$. This result (24) says that the optimal scheme lies within a class of schemes where the two-loop hadronic anomalous dimension vanishes, thus justifies the assumtion adopted in the analysis of the complete photon structure function especially in the large-n (or large-x) regions. It is also to be noted that in the optimal scheme corrections to the structure function coming from the constituent quark scattering vanish exactly, $B_{NS}=0$.

C. Optimization in the small coupling limit

Here we consider the structure function in the small coupling limit, Eq. (11). The response of $\hat{F}^{\gamma}$ to changes of scheme-labeling parameters become as follows (consistency equations (19) and (20) are used);

$$\frac{\partial \hat{F}^{\gamma}}{\partial \ln \mu} = -b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial \ln \mu} + b\delta_{NS}\, B_{NS}\, \tilde{a}^2(1+c\tilde{a})\,(1+ca)^{d^1_{NS}}$$
$$\times\left[\frac{K^0_{NS}}{1+d_{NS}}\,\frac{1+ca}{a} + \frac{K^1_{NS} - cK^0_{NS}(1+d^1_{NS})}{d_{NS}}\right.$$
$$\left. + \frac{c\, d^1_{NS}}{d_{NS}-1}\left\{\frac{1}{2}cK^0_{NS}(1+d^1_{NS}) - K^1_{NS}\right\}\frac{a}{1+ca}\right], \tag{25a}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial \ln M} = -b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial \ln M} + b\delta_{NS}(1+ca)^{d^1_{NS}}\left[\left\{-d_{NS}\,\tilde{a} + c\, d^1_{NS}(1+B_{NS}\,\tilde{a})\,a^2\right\}\right.$$
$$\times\left[\frac{K^0_{NS}}{1+d_{NS}}\,\frac{1+ca}{a} + \frac{K^1_{NS} - cK^0_{NS}(1+d^1_{NS})}{d_{NS}}\right.$$
$$\left. + \frac{cd^1_{NS}}{d_{NS}-1}\left\{\frac{1}{2}cK^0_{NS}(1+d^1_{NS}) - K^1_{NS}\right\}\frac{a}{1+ca}\right]$$
$$+ (1+B_{NS}\tilde{a})\,a\left[\frac{-K^0_{NS}}{1+d_{NS}}\,\frac{1+ca}{a} + \frac{cd^1_{NS}}{d_{NS}-1}\right.$$
$$\left.\left. \times\left\{\frac{1}{2}cK^0_{NS}(1+d^1_{NS}) - K^1_{NS}\right\}\frac{a}{1+ca}\right]\right], \tag{25b}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial d^1_{NS}} = -b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial d^1_{NS}} + \delta_{NS}(1+ca)^{d^1_{NS}}\left[\left\{-c\tilde{a} + (1+B_{NS}\tilde{a})\ln(1+ca)\right\}\right.$$
$$\times\left[\frac{K^0_{NS}}{1+d_{NS}}\,\frac{1+ca}{a} + \frac{K^1_{NS} - cK^0_{NS}(1+d^1_{NS})}{d_{NS}}\right.$$
$$\left. + \frac{cd^1_{NS}}{d_{NS}-1}\left\{\frac{1}{2}cK^0_{NS}(1+d^1_{NS}) - K^1_{NS}\right\}\frac{a}{1+ca}\right]$$
$$\left. + (1+B_{NS}\tilde{a})\left[-\frac{cK^0_{NS}}{d^{NS}} + \frac{c}{d_{NS}-1}\left\{\frac{c}{2}K^0_{NS}(1+2d^1_{NS}) - K^1_{NS}\right\}\frac{a}{1+ca}\right]\right], \tag{25c}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial K^1_{NS}} = -b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial K^1_{NS}} + \delta_{NS}(1+ca)^{d^1_{NS}}(1+B_{NS}\tilde{a})\left(\frac{1}{d_{NS}} - \frac{cd^1_{NS}}{d_{NS}-1}\,\frac{a}{1+ca}\right). \tag{25d}$$

Imposing the consistency requirement, namely, the variations (25a-d) to be order a, we get consistency equations;

$$\frac{\partial B_\Gamma}{\partial \ln \mu} = 0 , \tag{26a}$$

$$\frac{\partial B_\Gamma}{\partial \ln M} = -\frac{\delta_{NS}}{\delta_\Gamma} K^0_{NS} , \tag{26b}$$

$$\frac{\partial B_\Gamma}{\partial d^1_{NS}} = -\frac{c\delta_{NS} K^0_{NS}}{b\delta_\Gamma d_{NS}} , \tag{26c}$$

$$\frac{\partial B_\Gamma}{\partial K^1_{NS}} = \frac{\delta_{NS}}{b\delta_\Gamma d_{NS}} . \tag{26d}$$

Integrating the consistency equations for B_Γ, Eqs. (26), and those for B_{NS}, Eqs.(19) and (20), we get

$$B_\Gamma(Q/M) = \frac{\delta_{NS} K^0_{NS}}{b\delta_\Gamma d_{NS}} \left(d_{NS}\, b \ln\frac{Q}{M} - c d^1_{NS} + \frac{K^1_{NS}}{K^0_{NS}} \right) + \kappa_\Gamma , \tag{27a}$$

$$B_{NS}(Q/M) = d_{NS}\, b \ln\frac{Q}{M} - c d^1_{NS} + \kappa_{NS} , \tag{27b}$$

where κ_Γ and κ_{NS} are scheme-invariants calculable in any scheme, and Q denotes in general the large momentum scale inherent in the process considered (in the present case the four-momentun transfered from the current).

Finally we set the variations (25a-d) to be at most $\mathcal{O}(a^2)$. With the use of consistency equations (26a-d) we get the optimization equations,

$$B_{NS} = 0 , \tag{28a}$$

$$\frac{K^0_{NS}}{1+d_{NS}} \left(d_{NS}\, b \ln\frac{M}{\mu} + c d^1_{NS} - B_{NS} \right) - K^1_{NS} = 0 , \tag{28b}$$

$$\frac{K^0_{NS}}{1+d_{NS}} \left(b \ln\frac{M}{\mu} - \frac{c}{2} + B_{NS} \right) - \frac{K^0_{NS}}{d_{NS}} \left(c d^1_{NS} + B_{NS} \right) + \frac{1}{d_{NS}-1} \left\{ \frac{c}{2} K^0_{NS}(1 + 2d^1_{NS}) - K^0_{NS} \right\} = 0 , \tag{28c}$$

$$d^1_{NS} = 0 . \tag{28d}$$

It is worth noticing that the first equation (28a) in fact holds exactly, and the fourth equation (28d) holds up to the hadronic contributions of order $\mathcal{O}(a^{d_{NS}})$. Other equations (28b and c) hold up to the order-a-accuracy level. By solving the above equations, optimal values for the scheme-labeling parameters and for the perturbation coefficients are determined as follows;

$$\ln\frac{M}{\mu} = \frac{c}{b} , \tag{29a}$$

$$\ln\frac{Q}{M} = -\frac{\kappa_{NS}}{b d_{NS}} , \tag{29b}$$

$$K^1_{NS} = \frac{d_{NS}}{1+d_{NS}} c K^0_{NS} , \tag{29c}$$

$$d_{NS}^{1} = 0 , \tag{29d}$$

and

$$B_{NS} = 0 , \tag{30a}$$

$$B_{T} = \frac{\delta_{NS}}{b\delta_{T}} K_{NS}^{0} \left(b\ln\frac{Q}{M} + \frac{c}{1+d_{NS}} \right) + \kappa_{T} . \tag{30b}$$

As a result we get the optimized structure function as

$$\hat{F}^{T} = \frac{\delta_{NS} K_{NS}^{0}}{1+d_{NS}} \frac{1}{a} + \frac{\delta_{NS} K_{NS}^{0}}{d_{NS}} \left(\kappa_{NS} - \frac{c}{1+d_{NS}} \right) - b\delta_{T}\kappa_{T} , \tag{31}$$

where the optimal coupling constant a is given as a solution to the equation

$$\begin{aligned} \frac{1}{a} + c\ln\left(\frac{ca}{1+ca}\right) &= |b| \ln\frac{M}{\tilde{\Lambda}} \\ &= |b| \ln\frac{Q}{\tilde{\Lambda}} - \frac{\kappa_{NS}}{d_{NS}} . \end{aligned} \tag{32}$$

D. Optimization in the large-n limit

In order to see how the optimization based on the two-scale OPT has dealt with the large second order corrections for large-n moments that might be of kinematical origin, we here study the optimization in the large-n limit[22].

By keeping the leading terms in the large-n limit for quantities appearing in the optimization equations, we get the optimal value of the factorization scale $M_{opt}^{n} \equiv M_{opt}^{n}(Q)$ (in this subsection, the moment-index n is reproduced) as

$$\ln\left(\frac{M_{opt}^{n}}{Q}\right) \cong -\frac{1}{4}\ln n , \tag{33}$$

and the optimized coupling constant $a_{opt}^{n} \equiv a(M_{opt}^{n})$ as

$$\begin{aligned} a_{opt}^{n} &= \frac{1}{|b| \ln(M_{opt}^{n}/\tilde{\Lambda})} + \mathcal{O}\left(\frac{\ln\ln(M_{opt}^{n}/\tilde{\Lambda})}{\ln^{2}(M_{opt}^{n}/\tilde{\Lambda})}\right) \\ &\simeq \frac{1}{|b|\ln(Qn^{-1/4}/\tilde{\Lambda})} . \end{aligned} \tag{34}$$

Expressing the optimized structure function $\hat{F}_{n}^{T}(opt)$ as

$$\hat{F}_{n}^{T}(opt) = \frac{a_{n}}{a_{opt}^{n}} + b_{n}(opt) \tag{35}$$

we get(with the use of Eq.(31))

$$b_{n}(opt)/a_{n} \simeq \frac{2}{3}\ln n + \mathcal{O}(1) . \tag{36}$$

These results (33)-(36) are completely the same as those obtained in the kinematical analyses[12),15)] where kinamatically singular terms $a\ln^{2}n$ are resummed to all orders of the strong coupling constant a, and eventually absorbed into the rescaled effective mass M_{eff}^{n}, with which the effective coupling constant runs, $a_{eff}^{n} \equiv a(M_{eff}^{n})$. They show the same large-n behavior as Eqs.(33) and (34). As a result, the structure function calculated in terms of the rescaled coupling constant a_{eff}^{n} satisfies the

same expression as Eq. (35) with the perturbation coefficient $b_n(eff)$ showing the same large-n behavior as Eq. (36)[12]. With Eqs. (34) and (35) we can also show, by following the analysis by Frazer and Rossi[19], that the structure function in the x-space can be expressed as

$$\hat{F}^{\gamma}_{2,opt}(x, Q^2) \simeq h(x)/\ a((1-x)^{1/2}Q^2/\tilde{\Lambda}^2) + \tilde{h}(x) , \qquad (37)$$

where $\hat{F}^{\gamma}_0 = h(x)/a(Q^2/\tilde{\Lambda}^2)$ is the large-x form of the leading order result.

IV. Conclusions and Discussion

In this paper we applied the two-scale formalism of the optimized perturbation theory (OPT) to the second order QCD calculation of the photon structure function. In order to work out the complete optimization analytically without any assumptions, we consider, not the complete photon structure function, but a simplified two-component model that keeps the essential behavior of the complete structure function and in fact becomes exact in the large-n limit. The result shows that the two-scale formalism of OPT actually resolves completely the scheme-dependent ambiguities inherent in perturbative QCD calculation, thus giving an unique QCD prediction. This fact is to be compared with the analysis given in Ref. 7, where only the RS-dependence is taken into account and the optimization cannot be worked out for the photon structure function.

Analysis of the unique optimal result clarifies an another important observation. The OPT based on the principle of minimal sensitivity (PMS) has successfully dealt with the large second order corrections in the large-n moment, which may be of kinematical origin, precisely the same way as the kinematical approaches have done[12),15]. At present we have no definite idea to understand why such "equivalence" between the OPT and the kinematical analyses has held[23]. Only we can say now is that the OPT based on PMS has correctly taken into account the kinematical boundary effects, and that the optimized perturbation series shows an improved convergence behavior. This fact may justify the validity of the optimization based on PMS, giving it an another physical insight. A more detailed analysis concerning how the OPT has dealt with the (kinematically) singular terms in the perturbation coefficients has been given elsewhere[18].

Finally we take notice of the fact that, as was shown in section III B, the optimal scheme lies within a class of schemes where the two-loop hadronic anomalous dimension γ^1_{NS} vanishes. This fact *a posteriori* justifies the assumption utilized in the OPT analysis of the complete photon structure function given in Ref. 17, where optimization is carried out by restricting the parameter-space with the assumption that the optimal scheme satisfies the above condition.

References and Footnotes

1) E. Witten, Nucl. Phys. **B120**, 189 (1977).

2) W. A. Bardeen and A. J. Buras, Phys. Rev. **D20**, 166 (1979); D. W. Duke and J. F. Owens,

Phys. Rev. **D22**, 2280 (1980).

3) M. Glück and E. Reya, Phys. Rev. **D28**, 2749 (1984); M. Glück, K. Grassie and E. Reya, Phys. Rev. **D30**, 1447 (1984).

4) H. Nakkagawa and A. Niégawa, Prog. Theor. Phys. **71**, 339 (1984).

5) See, e. g., D. W. Duke and R. G. Roberts, Phys. Rep. **120**, 275 (1985), and references therein.

6) W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **42**, 1435 (1979); Phys. Rev. **D20**, 1420 (1979); W. Celmaster and D. Sivers, Phys. Rev. **D23**, 227 (1981).

7) P. M. Stevenson, Phys. Lett. **100B**, 61 (1981); Phys. Rev. **D23**, 2916 (1981).

8) W. Celmaster and D. Sivers, ANL-Reports ANL-HEP-PR-80-61 (1980), unpublished; Ann. Phys. (N. Y.) **143**, 1 (1983).

9) H. D. Politzer, Nucl. Phys. **B194**, 493 (1982).

10) H. Nakkagawa and A. Niégawa, Phys. Lett. **119B**, 415 (1982); Prog. Theor. Phys. **70**, 511 (1983).

11) The following is an incomplete sampling of references; the $x \to 1$ problem has been noted and remarked upon by so many workers that a complete citing is impossible. G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. **B143**, 521 (1978); **B146**, 544 (1978); **B157**, 461 (1979); J. Kubar-André and F. E. Paige, Phys. Rev. **D19**, 221 (1979).

12) W. Frazer and G. Rossi, Phys. Rev. **D25**, 843 (1982).

13) H. Nakkagawa and A. Niégawa, Prog. Theor. Phys. **71**, 816 (1984).

14) S. Brodsky and G. P. Lepage, in *Quantum Chromodynamics*, Proc. of the SLAC Summer Institute on Particle Physics, 1979, Ed. Anne Mosher (SLAC, Staford, 1980), p. 133.

15) D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. **B173**, 429 (1980).

16) M. Ciafaloni, Phys. Lett. **95B**, 113 (1980); M. Ciafaloni and G. Curci, Phys. Lett. **102B**, 352 (1981).

17) H. Nakkagawa, A. Niégawa and H. Yokota, Perturbative QCD analysis of the photon structure function, to be published in Phys. Rev. D.

18) H. Nakkagawa, A. Niégawa and H. Yokota, Large-n behavior of structure functions and the renormalization and factorization scheme dependences, Osaka City University preprint OCU-116 (1985).

19) We set, for simplicity but without loss of generality, the lower limit of the integration to be unity.

20) D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1323 (1973); Phys. Rev, **D8**, 3638 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); Phys. Rep. **14C**, 129 (1974).

21) E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. **B129**, 66 (1977); **B139**, 545 (1978); Phys. Lett, **80B**, 269 (1979); A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. **B166**, 429 (1980).

22) In order that the perturbation expansion in terms of the QCD effective coupling constant works, the large momentum scale Q is supposed to be large enough so as to satisfy $Qn^{-1/4} \gtrsim \tilde{\Lambda}$, see Eq.(34) below.

23) This "equivalence" holds only with respect to the rescaled mass parameter $M \simeq Qn^{-1/4}$. The kinematical consideration says nothing about the resolution of scheme-dependent ambiguities.